科目 高等微積分 類組別 902

＊請在答案卷內作答

參考用

In the solution sheet, indicate clearly the problem number, write clearly and logically.

1. (15%) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Show that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} \leq \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}.$$

2. (15%) A set $A \subseteq \mathbb{R}^n$ is said to be dense in $B \subseteq \mathbb{R}^n$ if $B \subseteq \mathrm{cl}(A)$, here $cl(A)$ denotes the closure of the set A (or you can use an ϵ-version of the definition of density if you know it). If A is dense in $\mathbb{R}^n$ and U is open, prove that $A \cap U$ is dense in U.

3. (20%) Let $(M,d),(N,d)$ be metric spaces and $K \subset M$ be a compact set. Supose $f : K \to N$ is continuous, prove that f is uniformly continuous on K. If you don't know what a metric space is, you can treat them as $\mathbb{R}^n$. However, your proof must make use of the general property of the compactness of K and the continuity of f.

4. a) (10%) Let (M,d) be a metric space and $f_k : M \to \mathbb{R}^n$ be a sequence of continuous function. Suppose f_k converge uniformly to f on M. Prove that f is also continuous on M.
   b) (5%) Give an example showing (and write the proof for it) that this is not true if the f_k are just continuous.

5. (10%) True or false? If you think the following statement is false, give a counter-example (and prove that your example works) and if you think that it is true, prove it. Let $A \subset \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$, $x_o \in A$, $\vec{n} \in \mathbb{R}^n$ is a unit vector. Suppose f is differentiable in every direction $\vec{n}$ at x_o, then f is differentiable at x_o.

6. a) (10%) Let $f(x,y)$ be a real-valued function on $\mathbb{R}^2$, f is of class C^1, $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and continuous. Show that $\dfrac{\partial^2 f}{\partial y \partial x}$ exists, and

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

   b) (5%) Give an example and provide computations for it to show that if none of $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are continuous, then $\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$.

7. (10%) Let $f : [a,b] \to \mathbb{R}^n$ be such that f has at most a finite number of discontinuities. Prove that f is Riemann integrable.