There are a total 22 problems. The first 3 problems are CALCULATION problems, and the rest of 19 problems are MULTIPLE choices. All of the MUTIPLE choices problems have only SINGLE correct answer. Please CLEARLY write your answers with associated problem numbers on the answer sheet. There will be NO partial credit given to the MULTIPLE CHOICE problems, anything that is not on the answer sheet will NOT be counted.

Problem 1. (10 %) Given a matrix

$$\bar{\bar{\mathbf{A}}}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 0 & a \end{bmatrix}, \tag{1}$$

where a is a real number. Find the *rank* of $\bar{\bar{\mathbf{A}}}_1$ and give the corresponding range of a.

Problem 2. (15 %) For the matrix

$$\bar{\bar{\mathbf{M}}}_1 = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}, \tag{2}$$

(i) (10 %) Find the *Eigenvalues* and *Eigenvectors* for $\bar{\bar{\mathbf{M}}}_1$.

(ii) (5 %) Show that there exists a matrix $\bar{\bar{\mathbf{J}}}$, which is similar to the matrix $\bar{\bar{\mathbf{M}}}_1$ in the **Jordon form**, i.e.,

$$\bar{\bar{\mathbf{J}}} = \bar{\bar{\mathbf{Q}}}^{-1}\bar{\bar{\mathbf{M}}}_1\bar{\bar{\mathbf{Q}}}. \tag{3}$$

Problem 3. (25 %) Let $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ denote the set of real $m \times n$ matrices and the set of real $n \times 1$ column vectors, respectively, and let $\mathbb{S}^n_+$ denote the set of real $n \times n$ symmetric positive semidefinite (PSD) matrices. This problem includes two parts as follows:

(i) (15 %) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with r singular values $\sigma_1, \ldots, \sigma_r$, $\mathbf{U}_r = [\mathbf{u}_1, \ldots, \mathbf{u}_r]$ consisting of the associated r left singular vectors and $\mathbf{V}_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$ consisting of the associated r right singular vectors.

(1) Find the range space (i.e., column space) of $\mathbf{A}$, denoted as $\mathcal{R}(\mathbf{A})$, and the row space of $\mathbf{A}$ (i.e., $\mathcal{R}(\mathbf{A}^T)$) where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$;

(2) find the projection matrix $\mathbf{P_A}$ such that $\mathcal{R}(\mathbf{A}) = \{\mathbf{P_A x} \mid \mathbf{x} \in \mathbb{R}^n\}$, and $\mathbf{P_{A^T}}$;

(3) find the eignevalues and eigenvectors of $\mathbf{AA}^T$.

(ii) (10 %) Suppose that $\mathbf{A}, \mathbf{X} \in \mathbb{S}^n_+$, and $\text{Tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$ (i.e., the sum of all the diagonal elements of $\mathbf{A}$).

(1) Is $\text{Tr}(\mathbf{AX}) \geq 0$ true?

(2) Is it true that if $\text{Tr}(\mathbf{AX}) = 0$, then $\mathbf{AX} = \mathbf{0} \in \mathbb{R}^{n\times n}$ (zero matrix)? Prove or disprove your answer.

Problem 4. (2.5 %) Which of the following statement is NOT true?

(i) If $A \subset B$, then $P(A) \leq P(B)$.

(ii) If $P(B) > 0$, then $P(A|B) \geq P(A)$.

(iii) $P(A \cap B) \geq P(A) + P(B) - 1$.

(iv) $P(A \cap B^c) = P(A \cup B) - P(B)$.

Problem 5. (2.5 %) We toss two fair coins simultaneously and independently. If the outcomes of the two coins are the same, we win; otherwise we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is false?

(i) Events A and B are independent.

(ii) Events A and C are not independent.

(iii) Events A and B are not conditionally independent given C.

(iv) The probability of winning is 1/2.

Problem 6. (2.5 %) Suppose X, Y and Z are three independent discrete random variables. Then X and $Y + Z$ are

(i) always

(ii) sometimes

(iii) never

independent.

Problem 7. (2.5 %) Consider two random variables X and Y, each taking values in $\{1, 2, 3\}$. Let their joint PMF be such that for any $1 \leq x, y \leq 3$,

$$P_{X,Y}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in \{(1,3), (2,1), (3,2)\}. \\ \textit{strictly positive}, & \text{otherwise.} \end{cases}$$

Then,

(i) X and Y can be independent or dependent depending upon the *strictly positive* values

(ii) X and Y are always independent

(iii) X and Y can never be independent

Problem 8. (2.5 %) We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that i-th urn is empty?

(i) $\left(1 - \frac{1}{m}\right)^n$

(ii) $\left(1 - \frac{1}{n}\right)^m$

(iii) $\binom{m}{n}\left(1-\frac{1}{n}\right)^m$

(iv) $\binom{n}{m}\left(1-\frac{1}{m}\right)^n$

Problem 9. (2.5 %) For a biased coin, the probability of "heads" is 1/3. Let h be the number of heads in five independent coin tosses. What is the probability P(first toss is a head | $h=1$ or $h=5$)?

(i) $\dfrac{\frac{1}{3}\left(\frac{2}{3}\right)^4}{5\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

(ii) $\dfrac{\frac{1}{3}\left(\frac{2}{3}\right)^4}{\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

(iii) $\dfrac{\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}{5\frac{1}{3}\left(\frac{2}{3}\right)^4+\left(\frac{1}{3}\right)^5}$

(iv) $\dfrac{1}{5}$

Problem 10. (2.5 %) A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the cases

(i) $\dfrac{\binom{48}{22}}{\binom{52}{26}}$

(ii) $\dfrac{4\binom{48}{22}}{\binom{52}{26}}$

(iii) $\dfrac{48!}{22!}\dfrac{52!}{26!}$

(iv) $\dfrac{4!\binom{48}{22}}{\binom{52}{26}}$

Problem 11. (2.5 %) To obtain a driving license, John needs to pass his driving test. Every time John takes a driving test, with probability 1/2, he will clear the test independent of his past. John failed his first test. Given this, let Y be the additional number of tests John takes before obtaining a license. Then,

(i) $E[Y] = 1$

(ii) $E[Y] = 2$

(iii) $E[Y] = 0$

Problem 12. (2.5 %) Let $X_i, 1 \leq i \leq 4$ be independent Bernoulli random variable each with mean $p = 0.1$. Let $X = \sum_{i=1}^{4} X_i$. That is, X is a Binomial random variable with parameters $n = 4$ and $p = 0.1$. Then,

(i) $E[X_1 \mid X = 2] = 0.1$

(ii) $E[X_1 \mid X = 2] = 0.5$

(iii) $E[X_1 \mid X = 2] = 0.25$

Problem 13. (2.5 %) Let X_1, X_2, X_3 be independent random variables with the continuous distribution over [0,1]. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

(i) 1/6

(ii) 1/3

(iii) 1/2

(iv) 1/4

Problem 14. (2.5 %) Let X and Y be two continuous random variables. Then,

(i) $E[XY] = E[X]E[Y]$

(ii) $E[X^2 + Y^2] = E[X^2] + E[Y^2]$

(iii) $f_{X+Y}(x+y) = f_X(x)f_Y(y)$

(iv) $var(X+Y) = var(X) + var(Y)$

Problem 15. (2.5 %) Suppose X is uniformly distributed over $[0,4]$ and Y is uniformly distributed over $[0,1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

(i) $f_Z(4.5) = 0$

(ii) $f_Z(4.5) = 1/8$

(iii) $f_Z(4.5) = 1/4$

(iv) $f_Z(4.5) = 1/2$

Problem 16. (2.5 %) For the random variables defined in problem 13, $\mathbf{P}(\max(X,Y)>3)$ is equal to

(i) 0

(ii) 9/4

(iii) 3/4

(iv) 1/4

Problem 17. (2.5 %) N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $E[X]$ is equal to

(i) λ

(ii) $\frac{1}{\lambda^2}$

(iii) $\frac{1}{\lambda}$

(iv) 1

Problem 18. (3.0 %) Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then,

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) W is Poisson with parameter $\lambda_1 + \lambda_2$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

Problem 19. (3.0 %) Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then,

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\mathbf{P}(X = 0) = 0.4^{50}$

(iv) $\mathbf{P}(X = 50) = 0.6$

Problem 20. (3.0 %) Let $X_i, i = 1, 2, ...$ be independent random variables all distributed according to the pdf $f_X(x) = x/8$ for $0 \leq x \leq 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $1 - \Phi\left(\frac{5}{\sqrt{2}}\right)$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

Problem 21. (3.0 %) Let $X_i, i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = 1$ for $0 \leq x \leq 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$ for some integer n. Then $var(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\frac{1}{3^n} - \frac{1}{4^n}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Problem 22. (3.0 %) Suppose you play a matching coin game with your friend as follows. Both you and your friend have a coin. Each time, you two reveal a side (i.e. H or T) of your coin to each other simultaneously. If the sides match, you WIN a 1 from your friend if sides do not match then you lose a 1 your friend. You decide to go with the following strategy. Every time, you will toss your unbiased coin independently of everything else, and you will reveal its outcome to your friend (your friend does not know the outcome of your random toss until you reveal it). Then,

(i) On average, you will lose money to your smart friend

(ii) On average, you will lose neither lose nor win. That is, your average gain/loss is 0.

(iii) On average, you will make money from your friend