- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題五分。每題ABCDE每一選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分。倒扣至該大題0分為止。

Notation: In the following questions, underlined letters such as $\underline{a}$, $\underline{b}$, etc. denote column vectors of proper length; boldface letters such as $\mathbf{A}, \mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\|\underline{a}\|$ means the Euclidean norm of vector $\underline{a}$. $\mathbb{R}$ is the usual set of all real numbers. $\det(\mathbf{A})$ is the determinant of square matrix $\mathbf{A}$. $\text{row}(\mathbf{A})$ and $\text{col}(\mathbf{A})$ are the row and column spaces of $\mathbf{A}$ over $\mathbb{R}$, respectively. For any linear map T over vector spaces, we use $\ker(T)$, $\text{rank}(T)$ and $\text{nullity}(T)$ for the kernel, rank and nullity of T, respectively. Let W be a subspace of $\mathbb{R}^n$; then by $W^\perp$ we mean the orthogonal complement of W in the Euclidean inner product space $\mathbb{R}^n$. $\mathscr{L} : f(t) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively.

一、 Which of the following sets is basis (are bases) for $\mathbb{R}^3$?

(A) $\{[0,1,0]^\top,\ [0,0,1]^\top,\ [1,0,0]^\top\}$.

(B) $\{-2,4,-6]^\top,\ [1,-2,3]^\top\}$.

(C) $\{[0.14,0,-0.1]^\top,\ [-1,-0.2,0.4]^\top,\ [0.5,0.5,-1]^\top\}$.

(D) $\{[-1,2,3]^\top,\ [8,-7,6]^\top,\ [4,-2,3]^\top, [9,0,5]^\top\}$.

(E) None of the above are true.

二、Which of the following statements about the multiplicative inverse of a matrix is/are true?

(A) A matrix **A** is called invertible if there exists a matrix **B** such that **AB** is an identity matrix.

(B) If a matrix is both diagonalizable and invertible, then so is its multiplicative inverse.

(C) Suppose that matrices **A** of size $n \times n$ and **D** of size $m \times m$ are invertible and that matrix **C** is of size $m \times n$; then the following identity is true

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{D}^{-1} \end{bmatrix}.$$

(D) Methods for finding the multiplicative inverse of a matrix include LU factorization, Gaussian elimination, eigen-decomposition, and Gram-Schmidt process.

(E) None of the above are true.

三、Which of the following properties of eigenvalue is/are true?

(A) A scalar λ is an eigenvalue of matrix **A** if and only if λ is an eigenvalue of $\mathbf{A}^\top$.

(B) A matrix is positive semi-definite if and only if all of its eigenvalues are non-negative.

(C) Every eigenvalue of a matrix **A** is also an eigenvalue of $\mathbf{A}^2$.

(D) If matrices **A** and **B** are similar, then they have the same eigenvalues.

(E) None of the above are true.

四、Which of the following matrices is/are diagonalizable?

(A) $\begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$.

(B) $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(C) $\begin{bmatrix} -2 & 8 & -4 \\ -6 & 8 & 0 \\ -6 & 2 & 6 \end{bmatrix}$.

(D) $\begin{bmatrix} 2 & 0 & -2 & 9 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

(E) None of the above are true.

五、Which of the following statements about matrix factorization is/are true?

(A) If a matrix $\mathbf{A}$ is positive definite, then $\mathbf{A}$ has "an" LU factorization, $\mathbf{A} = \mathbf{LU}$, where the diagonal entries of $\mathbf{U}$ are positive.

(B) Suppose that a matrix $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an $m \times n$ matrix and $\mathbf{R}$ is an $n \times n$ matrix. If the columns of $\mathbf{A}$ are linearly independent, then $\mathbf{R}$ must be invertible.

(C) Any factorization of a matrix $\mathbf{A} = \mathbf{UDV}^\top$, with matrices $\mathbf{U}$, $\mathbf{V}$ square and positive diagonal entries in the matrix $\mathbf{D}$, is called a singular value decomposition of $\mathbf{A}$.

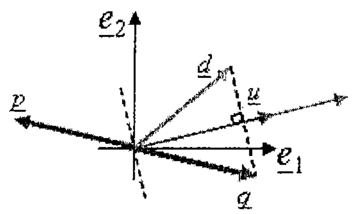
(D) An $n \times n$ matrix $\mathbf{A}$ is positive definite if and only if $\mathbf{A}$ has "a" Cholesky factorization $\mathbf{A} = \mathbf{R}^\top\mathbf{R}$ for some invertible upper triangular matrix $\mathbf{R}$ whose diagonal entries are all positive.

(E) None of the above are true.

注意:背面有試題

六、The Householder matrix $\mathbf{H} = \mathbf{I}_n - 2\text{proj}_{\underline{u}}$, where $\text{proj}_{\underline{u}} = \frac{1}{\|\underline{u}\|^2}\underline{u}\underline{u}^\top$ is the orthogonal projection matrix onto some nonzero vector $\underline{u} \in \mathbb{R}^n$, is a reflection matrix. Which of the following statements is/are true?

(A) Consider the 2-dimensional space, i.e. $n = 2$. For vectors $\underline{u}, \underline{d}, \underline{p}, \underline{q} \in \mathbb{R}^2$ shown in the figure below, we have $\mathbf{H}\,\underline{d} = \underline{p}$.



(B) $\mathbf{H}$ is a symmetric and orthogonal matrix.

(C) Both linear systems $\mathbf{A}\,\underline{x} = \underline{b}$ and $\mathbf{HA}\,\underline{x} = \mathbf{H}\,\underline{b}$ are equivalent.

(D) Let $\underline{a} = [a_1, \ldots, a_n]^\top$ and $\underline{u} = [a_1 - \|\underline{a}\|, a_2, \ldots, a_n]^\top \neq \underline{0}$. Then $\|\underline{u}\|^2 = -2\,\|\underline{a}\|\,u_1$ and $\mathbf{H}\,\underline{a} = [\|\underline{a}\|, 0, \ldots, 0]^\top$.

(E) None of the above are true.

七、 Given

$$\mathbf{A} = \begin{bmatrix} 1 & 11 & 23 & 81 & 97 \\ 2 & 22 & 46 & 162 & 194 \\ 3 & 1 & 1 & 11 & 2 \\ 9 & 0 & 1 & -4 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 7 & 6 & 0 & 0 & 0 \\ 3 & 9 & 2 & 0 & 0 \\ 3 & -4 & 0 & -1 & 0 \\ 1 & 2 & 101 & -5 & -2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 7 & 1 & 0 & 5 & 37 \\ 3 & 20 & 0 & 9 & 71 \\ 8 & -71 & 0 & 1 & 13 \\ 2 & -2 & 0 & 2 & 3 \\ 1 & -5 & 0 & 45 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 4 & 0 & -2 \\ 37 & 20 & 7 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -10 & -1 & 0 & -2 \\ 2 & -5 & -94 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 7 & 1 & 23 & 0 & 0 & 1 \\ 1 & 2 & 101 & -5 & -2 & 0 \\ 3 & -4 & 0 & -1 & 0 & 0 \\ 3 & 9 & 2 & 0 & 0 & 0 \\ 7 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 2 & 9 \\ 7 & 2 & 0 & 3 & 7 \\ 3 & 7 & 1 & 5 & -9 \\ 0 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix},$$

which of the following statements is/are true?

(A) $\det(\mathbf{A}) = \det(\mathbf{C}) = 0$.

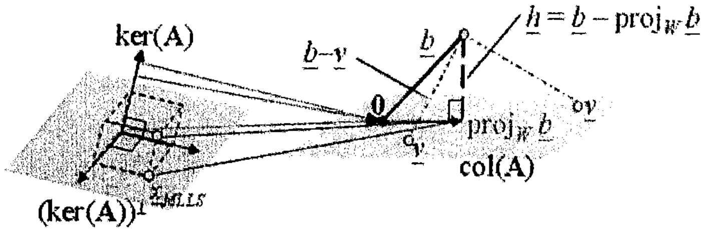
(B) $\det(\mathbf{D}) = -40$.

(C) $\det(\mathbf{B}) = \det(\mathbf{E})$.

(D) $\det(\mathbf{F}) = 20$.

(E) None of the above are true.

八、Consider a 3×3 nonzero matrix $\mathbf{A}$, and let $W = \text{col}(\mathbf{A})$ and $V = \text{col}(\mathbf{A}^\top)$. The least squares solutions $\underline{x}_{\text{LS}}$ to $\mathbf{A}\underline{x} = \underline{b}$ are illustrated in the following figure, where $\text{proj}_W \underline{b}$ is the orthogonal projection of vector $\underline{b}$ onto vector space W. A minimum length least squares solution $\underline{x}_{\text{MLLS}}$ is the one among the least squares solutions that has a minimum norm.



Which of the following statements is/are true?

(A) $(\text{col}(\mathbf{A}))^\perp = \ker(\mathbf{A}^\top)$ is proved by either (1) if $\underline{z} \in (\text{col}(\mathbf{A}))^\perp$, then $\underline{z}^\top \mathbf{A}\underline{x} = 0$ for any $\underline{x}$; and then $\mathbf{A}^\top \underline{z} = \underline{0}$; or (2) if $\underline{s} \in \ker(\mathbf{A}^\top)$, then $\underline{s}^\top \mathbf{A}\underline{x} = 0$ for any $\underline{x}$.

(B) Let $\underline{h} = \underline{b} - \text{proj}_W \underline{b}$ (see the above figure). Then $\mathbf{A}^\top \underline{h} = \underline{0}$ and the system $\mathbf{A}\underline{x} = \underline{h} + \underline{v}$ for any $\underline{v} \in \text{col}(\mathbf{A})$ is a consistent system, i.e., $\mathbf{A}\underline{x} = \underline{h} + \underline{v}$ has at least one solution (not least squares solution).

(C) A least squares solution $\underline{x}_{\text{LS}}$ to $\mathbf{A}\underline{x} = \underline{b}$ satisfies $\mathbf{A}\underline{x} = \text{proj}_W \underline{b}$, and then satisfies $\mathbf{A}^\top \mathbf{A}\underline{x} = \mathbf{A}^\top \underline{b}$.

(D) The minimum length least squares solution $\underline{x}_{\text{MLLS}}$ to $\mathbf{A}\underline{x} = \underline{b}$ is unique, and $\underline{x}_{\text{MLLS}} = \text{proj}_V \underline{x}_{\text{LS}}$, where $V = \text{col}(\mathbf{A}^\top)$.

(E) None of the above are true.

九、 Let $\underline{v}_1$ and $\underline{v}_2$ be the eigenvectors of matrix $\mathbf{A}$ corresponding respectively to eigenvalues λ_1 and λ_2, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \ \underline{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

It is found that $\lambda_1 > 1 > |\lambda_2|$. With matrix $\mathbf{A}$, let $\underline{x}_n = [x_{n,1}, x_{n,2}]^\top \in \mathbb{R}^2$ for $n = 0, 1, \ldots$ be a series of vectors related by $\underline{x}_{n+1} = \mathbf{A}\underline{x}_n$. Given the initial condition $\underline{x}_0 = [1, 0]^\top = \alpha\underline{v}_1 + \beta\underline{v}_2$ for some $\alpha, \beta \in \mathbb{R}$, which of the following statements is/are true?

(A) $\beta = \frac{1}{\lambda_1 - \lambda_2}$.

(B) $\underline{x}_n = \alpha(\lambda_1)^n\underline{v}_1 + \beta(\lambda_2)^n\underline{v}_2$.

(C) $\lim_{n\to\infty} \frac{x_{n,1}}{x_{n-1,1}} = \lambda_1$.

(D) $\lim_{n\to\infty} \frac{x_{n,2}}{x_{n-1,2}} = \lambda_1$.

(E) None of the above are true.

十、 Let $T_{\underline{u}}$ be a linear transformation on $\mathbb{R}^3$ for a rotation by an angle θ about a unit vector $\underline{u}$. Specifically, we let the matrix for $T_{\underline{u}}$ with respect to the standard basis S for $\mathbb{R}^3$ be

$$\mathbf{G} = [T_{\underline{u}}]_S = \begin{bmatrix} c + u_1^2(1-c) & u_1u_2(1-c) - u_3s & u_1u_3(1-c) + u_2s \\ u_1u_2(1-c) + u_3s & c + u_2^2(1-c) & u_2u_3(1-c) - u_1s \\ u_1u_3(1-c) - u_2s & u_2u_3(1-c) + u_1s & c + u_3^2(1-c) \end{bmatrix},$$

where $[\underline{u}]_S = [u_1, u_2, u_3]^\top$ is the coordinate vector of $\underline{u}$ with respect to S, $c = \cos(\theta)$ and $s = \sin(\theta)$. Furthermore, let $\mathbf{A}$ be the rotation matrix about the z-axis of Cartesian coordinate system by an angle θ, i.e.,

$$\mathbf{A} = [T_{\underline{v}}]_S = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $[\underline{v}]_S = [0, 0, 1]^\top$. Let $B = \{\underline{n}, \underline{b}, \underline{u}\}$ be an ordered orthonormal basis for $\mathbb{R}^3$ with $[\underline{n}]_S = [n_1, n_2, n_3]^\top$ and $[\underline{b}]_S = [b_1, b_2, b_3]^\top$ and let $\mathbf{P}_{S \leftarrow B} = [[\underline{n}]_S, [\underline{b}]_S, [\underline{u}]_S]$ be the change-of-basis matrix for changing basis from B to S. Which of the following statements is/are true?

(A) $n_1^2 + b_1^2 + u_1^2 = 1$, $n_1n_2 + b_1b_2 + u_1u_2 = 0$ and $n_1n_3 + b_1b_3 + u_1u_3 = 0$.

(B) The coordinate vector of $\underline{u}$ with respect to basis B is $[\underline{u}]_B = \mathbf{P}_{S \leftarrow B}\, [\underline{u}]_S$.

(C) $\mathbf{G} = \mathbf{P}_{S \leftarrow B}\mathbf{A}$.

(D) $\mathbf{G} = \mathbf{A}\mathbf{P}_{S \leftarrow B}$.

(E) The matrix for $T_{\underline{u}}$ with respect to basis B is $\mathbf{A}$.

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十一、 Solve for $y(x)$ the first order differential equation

$$xy'(x) - 4x^2y(x) + 2y(x)\ln(y(x)) = 0$$

by the substitution $v = \ln(y(x))$. Which of the following statements is/are true?

(A) It is a nonlinear ordinary differential equation for the dependent variable y.

(B) It is a nonlinear ordinary differential equation for the new variable v.

(C) There exists a solution $y(x)$ satisfying the condition $y(0) = 1$.

(D) There exists a solution $y(x)$ satisfying the condition $y(1) = 1$.

(E) None of the above are true.

十二、 The Cauchy-Euler equation

$$x^2y''(x) - 4xy'(x) + 6y(x) = 0$$

can be transformed into a constant coefficient equation $y''(v) + by'(v) + cy(v) = 0$ by the substitution $v = \ln(x)$. Which of the following statements is/are true?

(A) $b = -5$.

(B) $c = -6$.

(C) The solution $y(x)$ exists only for $x > 0$.

(D) $y(x) = C_1x^2 + C_2x^3$ for some constants C_1 and C_2.

(E) None of the above are true.

十三、 Continued from Problem 十二. Solve the non-homogeneous second order differential equation

$$x^2y''(x) - 4xy'(x) + 6y(x) = x^3$$

by variation of parameters, i.e., set the particular solution as $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where $y_1(x)$ and $y_2(x)$ are homogeneous solutions. With initial conditions $y(1) = 0$ and $y'(1) = 1$ for the complete solution $y(x)$, which of the following statements is/are true?

(A) The real valued solution $y(x)$ exists only for $x > 0$.

(B) $y(2) = 2$.

(C) $2y'(2) - 3y(2) = 8$.

(D) $y'(3) - y(3) = 4$.

(E) None of the above are true.

十四、The first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8 & 0 & 17 & 0 \\ 0 & 0 & 0 & 1 \\ 17 & 0 & 8 & 0 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

can be reduced into an equivalent second order system $\underline{y}''(t) = \mathbf{B}\,\underline{y}(t)$ with $\underline{y}(t) = [x_1(t), x_3(t)]^\top$. Which of the following statements is/are true?

(A) $\mathbf{B} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$.

(B) $\mathbf{B} = \begin{bmatrix} 8 & 17 \\ 17 & 8 \end{bmatrix}$.

(C) The eigenvalues of $\mathbf{B}$ are 9 and 25.

(D) $[1, 1]^\top$ is an eigenvector for $\mathbf{B}$.

(E) None of the above are true.

十五、 Continued from Problem 十四. Find the particular solution for the second order system $\underline{y}''(t) = \mathbf{B}\,\underline{y}(t)$ with initial conditions $\underline{y}(0) = [1,3]^\top$ and $\underline{y}'(0) = [-3,3]^\top$. Which of the following statements is/are true?

(A) $x_1(t) = e^{-5t} + e^{5t} - \cos(3t) - \sin(3t)$.

(B) $x_3(t) = e^{-5t} + e^{5t} + e^{3t}$.

(C) $(x_1(t) + x_3(t))$ is an odd function in t.

(D) $(x_1(t) - x_3(t))$ is an odd function in t.

(E) None of the above are true .

十六、 For $s > 0$, let $F(s)$ be the unilateral Laplace transform of function $f(t)$ given by

$$F(s) = \frac{1}{2s^2} - \frac{1}{s(e^s + e^{3s})}.$$

Which of the following statements regarding the values of $f(t)$ is/are true?

(A) $f(1) = \frac{1}{2}$.

(B) $f(2) = 1$.

(C) $f(4) = 2$.

(D) $f(8) = 3$.

(E) None of the above are true.

十七、 Consider the following periodic function

$$f(t) = \sum_{n=-\infty}^{\infty} \exp\left(-\pi(t-n)^2\right)$$

which has a Fourier series representation

$$f(t) = \sum_{m\geq 0} a_m \cos(2\pi mt) + b_m \sin(2\pi mt)$$

for some $a_m, b_m \in \mathbb{R}$ and for all $t \in \mathbb{R}$. Which of the following statements is/are true?

(A) $a_0 = \frac{1}{\sqrt{\pi}}$.

(B) $a_1 = e^{-\pi}$.

(C) $a_2 = \frac{-3}{5\pi}$.

(D) $b_2 = e^{-4\pi}$.

(E) None of the above are true.

十八、 For the following second order differential equation

$$tx''(t) + (4t-2)x'(t) + (2t-4)x(t) = 0$$

Let $x_1(t) = t^{r_1}\sum_{n\geq 0} a_n t^n$ and $x_2(x) = t^{r_2}\sum_{n\geq 0} b_n t^n$ be the two linearly independent Frobenius series solutions for $x(t)$ when $x > 0$, where r_1 and r_2 are the zeros of the corresponding indicial equation. Assume $r_1 \geq r_2$ and $a_0 = b_0 = 1$. Which of the following statements is/are true?

(A) $r_1 - r_2$ is not an integer.

(B) $a_2 = \frac{13}{5}$.

(C) $a_3 = -\frac{6}{15}$.

(D) $b_3 = \frac{4}{3}$.

(E) None of the above are true.

十九、 Continued from Problem 十八. The second order differential equation can be alternatively solved by using Laplace transform. Assuming $x(0) = 0$ and $\int_0^\infty x(t)dt = 1$, which of the following statements is/are true about the values of $x(t)$ and its unilateral Laplace transform $X(s) = \mathscr{L}\{x(t)\}$?

(A) $x'(0) = 1$.

(B) $x(1) = 1$.

(C) $X(1) < 1$.

(D) Values of $X(s)$ exists for all $s > -1$.

(E) None of the above are true.

二十、Consider the following boundary value problem for the bivariate function $u(x,t)$ defined for $0 \le x \le \pi$ and $t \ge 0$

$$\frac{\partial}{\partial t}u(x,t) = 2\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)$$

Given the end-point and initial conditions

$$\left.\frac{\partial}{\partial x}u(x,t)\right|_{x=0} = \left.\frac{\partial}{\partial x}u(x,t)\right|_{x=\pi} = 0 \quad \text{and} \quad u(x,0) = x(\pi - x)$$

which of the following statements is/are true for the solution $u(x,t)$ when it is expressed as

$$u(x,t) = \sum_{n\ge 0} a_n e^{p_n t}\cos(2nx) + b_n e^{q_n t}\sin(2nx)$$

for some constants $a_n, b_n, p_n, q_n \in \mathbb{R}$?

(A) $a_0 = \frac{\pi}{6}$.

(B) $a_1 = -1$.

(C) $p_2 = -31$.

(D) $b_1 = -1$

(E) None of the above are true.