類組：電機類　科目：工程數學B(3004)　　共 10 頁 第 1 頁

※請在答案卡內作答

- 本測驗試題為多選題(答案可能有一個或多個)，請選出所有正確或最適當的答案，並請用 2B 鉛筆作答於答案卡。
- 共二十題，每題五分。每題 ABCDE 每一選項單獨計分。每一選項的個別分數為一分，答錯倒扣一分。

一、Consider a linear system $\begin{matrix} x_1 + 4x_2 = 3 \\ 3x_1 + hx_2 = k \end{matrix}$. Which of the following statements is/are true?

(A) When $h = 12$ and $k = 9$, the system is inconsistent.

(B) When $h = 12$ and $k = 9$, the system has many solutions.

(C) When $h = 12$ and $k \neq 9$, the system is consistent.

(D) When $h = 12$ and $k \neq 9$, the system has at least one solution.

(E) When $h \neq 12$, the system has a unique solution.

二、Denote det A as the determinant of the matrix A, and denote A^{-1} as the inverse of the matrix A. Let A, B, and P be square matrices. Which of the following statements is/are true?

(A) It is always true that $\det AB = \det BA$.

(B) If the columns of A are linearly dependent, then $\det A = 0$.

(C) It is always true that $\det(A + B) = \det A + \det B$.

(D) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

(E) Suppose that P is invertible. Then $\det(PAP^{-1}) = \det A$.

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三、Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Which of the following statements is/are true?

(A) The image of $\mathbf{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ under T is $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

(B) There is exactly one $\mathbf{x}$ whose image under T is $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.

(C) The vector $\mathbf{b}$ is in the range of T if $\mathbf{b}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$.

(D) The vector $\mathbf{b}$ is in the range of T if the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(E) The vector $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ is not in the range of T.

四、The set $\mathbb{P}_n$ of polynomials of degree at most n, $n \geq 0$, consists of all polynomials of the form $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$. Let $\mathbf{p}_1(t) = 2 + 2t^2$, $\mathbf{p}_2(t) = -t + 3t^2$, and $\mathbf{p}_3(t) = 1 + t - 3t^2$. Which of the following statements is/are true?

(A) $\mathbf{p}_1(t)$ is in $\mathbb{P}_3$.

(B) To see whether the polynomials $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, and $\mathbf{p}_3(t)$ form a basis for $\mathbb{P}_2$, we can place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form. If the resulting matrix is not invertible, then the polynomials $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, and $\mathbf{p}_3(t)$ form a basis for $\mathbb{P}_2$.

(C) Polynomials $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$, and $\mathbf{p}_3(t)$ form a basis for $\mathbb{P}_2$.

(D) Consider the basis $\mathfrak{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for $\mathbb{P}_2$. Let the vector $[\mathbf{q}]_{\mathfrak{B}}$ be the $\mathfrak{B}$-coordinate vector of $\mathbf{q}$. Given that $[\mathbf{q}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{q}$ in $\mathbb{P}_2$ is $2\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3$.

(E) Following (D), $\mathbf{q}(t) = 2 - t + t^2$.

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五、Suppose that a 6×3 matrix A has rank 3. Denote dim H as the dimension of a nonzero subspace H, Nul A as the null space of the matrix A, Row A as the column space of the matrix A, rank A as the rank of the matrix A, and A^T as the transpose of the matrix A. Which of the following statements is/are true?

(A) dim Row A = rank A.

(B) dim Row A = rank A^T.

(C) dim Nul A = 3.

(D) dim Row A = 6.

(E) rank A^T = 3.

六、Let $\mathbf{y}=\begin{bmatrix}7\\1\\-1\\-1\end{bmatrix}$, $\mathbf{v}_1=\begin{bmatrix}1\\2\\1\\-2\end{bmatrix}$, and $\mathbf{v}_2=\begin{bmatrix}4\\1\\0\\3\end{bmatrix}$. Let the subspace W be spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Which of the following statements is/are true?

(A) $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

(B) The closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\begin{bmatrix}5\\3\\1\\1\end{bmatrix}$.

(C) The closest point in W to $\mathbf{y}$ is the projection of $\mathbf{y}$ on W.

(D) The distance from the point $\mathbf{y}$ in $\mathbb{R}^4$ to W is defined as the distance from $\mathbf{y}$ to the closest point in W.

(E) The distance from $\mathbf{y}$ to the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is 16.

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七、Let A^{-1} be the inverse of the matrix A, A^T be the transpose of the matrix A, and I be the identity matrix. Which of the following statements about invertible matrices is/are true?

(A) If A is both diagonalizable and invertible, then so is A^{-1}.

(B) Suppose that $A = QR$, where R is an invertible matrix. Then A and Q have the same column space.

(C) Suppose that A and B are square matrices, B is invertible, and AB is invertible. Then A is invertible.

(D) Suppose that $A = PDQ^T$, where P and Q are $n\times n$ matrices with the property that $P^TP = I$ and $Q^TQ = I$, and D is a diagonal matrix with positive $\sigma_1,\ldots,\sigma_n$ on the diagonal. Then A is invertible.

(E) Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$, where $A_{11}, A_{12}, A_{21}, A_{22}, X$, and Y are matrices, and $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur complement of A_{11}. Suppose that A is invertible and A_{11} is invertible, then S is invertible.

八、Let $A = \begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$. Giving an orthogonal matrix P and a diagonal matrix D, we can orthogonally diagonalize A in the way $A = PDP^{-1}$, where P^{-1} is the inverse of P. Which of the following statements is/are true?

(A) The smallest eigenvalue of A is -3.

(B) For the smallest eigenvalue of A, a basis for the eigenspace is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. For the largest eigenvalue of A, a basis for the eigenspace is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. For the third eigenvalue of A, a basis for the eigenspace is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(C) P can be constructed as $P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$.

(D) D can be constructed as $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are the eigenvalues of A.

(E) $P^{-1} = P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, where P^T is the transpose of P.

九、Let the matrix $A = \begin{bmatrix} -3 & 2 \\ 6 & -4 \\ 6 & -4 \end{bmatrix}$. Let λ be the eigenvalue of A, and A^T be the transpose of A. Which of the following statements is/are true?

(A) The characteristic polynomial of A^TA is $\lambda^2 - 117\lambda$.

(B) The singular values of A are 117 and 0.

(C) Any factorization $A = U\Sigma V^T$, with U and V orthogonal, $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, and positive diagonal entries in D, is called a singular value decomposition (SVD) of A.

(D) Following (C), the matrices U and V are uniquely determined by A, and the diagonal entries of Σ are the singular values of A.

(E) An SVD of A is $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3\sqrt{13} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{13} & 2/\sqrt{13} \\ -2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$.

十、Which of the following statements about the properties of matrices is/are true?

(A) An $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly dependent.

(B) Suppose that A is an $m \times n$ matrix such that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Then the columns of A must be linearly dependent.

(C) λ is an eigenvalue of the matrix A if and only if λ is an eigenvalue of A^{-1}, the inverse of A.

(D) If the matrices A and B are both orthogonally diagonalizable and $AB = BA$, then AB is also orthogonally diagonalizable.

(E) The trace of a square matrix A, denoted by tr A, is the product of the diagonal entries in A. tr (FG) = tr (GF) for any two $n \times n$ matrices F and G.

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十一. Let X be a continuous random variable that is uniformly distributed over $(0,1)$ and $Y=-\ln(X)$. Which of the following statements is/are true?

(A) $P(X>0.25)=0.75$.

(B) $\mathrm{E}[X^2]=\frac{2}{3}$.

(C) $P(25X^2-20X+3>0)>0.55$.

(D) $\mathrm{E}[Y]=2$.

(E) $\mathrm{E}[Y^2]=2$.

十二. Let X be a Gaussian random variable such that $\mathrm{E}[X]=1$ and $\mathrm{E}[(X-1)^2]=4$. Let Y be a Gaussian random variable such that $\mathrm{E}[Y]=0$ and $\mathrm{E}[Y^2]=1$. In addition, X and Y are statistically independent. Let Z be a random variable such that $Z=2X+3Y$. Which of the following statements is/are true?

(A) $\mathrm{E}[Z]=2$.

(B) $\mathrm{E}[Z^2]=25$.

(C) $\mathrm{E}[YZ]=4$.

(D) Y^2 is an exponential random variable.

(E) $P(Z\geq 2)=\frac{1}{2}$.

十三. Let X be a continuous random variable with probability density function f_X. In addition, $f_X(x)=\frac{1}{2\sqrt{x}}$, $\forall x\in(0,1]$. Furthermore, $P(X>1)=P(X\leq 0)=0$. Moreover, $f_X(t)<\infty$, $\forall t\in(-\infty,\infty)$. Which of the following statements is/are true?

(A) $P(X=\frac{4}{9})=\frac{3}{4}$.

(B) $\int_0^2 f_X(x)\,dx=\sqrt{2}$.

(C) $\int_0^1 f_X(x)\,dx=1$.

(D) $\mathrm{E}[X]=\frac{2}{3}$.

(E) $P(X\in[\frac{1}{4},\frac{9}{16}])=\frac{1}{6}$.

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十四. Let $p \in (0,1)$ be a positive real number. Let X be a geometric random variable with PMF $p_X(k) = (1-p)^{k-1} \cdot p$, $\forall k \in \{1,2,3,...\}$. Which of the following statements is/are true?

(A) When $p = 0.6$, $P(X > 1) = 0.4$.

(B) $\sum_{k=3}^{\infty}(1-p)^{k-1} \cdot p = 1 - 2p + p^2$.

(C) When $p = 0.8$, $\mathrm{E}[X] = 5$.

(D) When $p = 0.5$, $\mathrm{E}[X|X > 1] = 3$.

(E) When $p = 0.5$, $P(X > 4|X > 1) = \frac{1}{16}$.

十五. Consider the famous hat problem. Suppose that n people throw their hats in a box and then each picks one hat at random. (Each hat can be picked by only one person, and each assignment of hats to persons is equally likely.) Let X be a random variable that represents the number of people that get back their own hat. Which of the following statements is/are true?

(A) When $n = 3$, $P(X = 3) = \frac{1}{3}$.

(B) When $n = 3$, $P(X = 1) = \frac{1}{2}$.

(C) When $n = 4$, $P(X = 3) = 0$.

(D) When $n = 4$, $\mathrm{E}[X] = 2$.

(E) When $n = 5$, $\mathrm{E}[X] = 1$.

十六. Consider the famous hat problem as in the previous problem. Let X_i be a random variable that takes value 1 if the ith person selects his/her own hat, and takes value 0 otherwise. Let $\mathrm{cov}(Y,Z) = \mathrm{E}[Y \cdot Z] - \mathrm{E}[Y] \cdot \mathrm{E}[Z]$ be the covariance of two random variables Y and Z. Which of the following statements is/are true?

(A) When $n = 4$, $\mathrm{cov}(X_4, X_4)$ is $\frac{3}{16}$.

(B) When $n = 4$, $\mathrm{cov}(X_1, X_2) = \frac{1}{48}$.

(C) When $n = 4$, $\mathrm{var}(\sum_{k=1}^{4} X_k) = 2$.

(D) When $n = 4$, X_1 and X_2 are statistically independent.

(E) For any two random variables Y and Z, $\mathrm{cov}(Y,Z) \geq 0$.

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十七. Consider a continuous random variable X. Let $M_X(s) = \mathrm{E}[e^{sX}]$, $\forall s \in (-\infty, \infty)$. Which of the following statements is/are true?

(A) If X is an exponential random variable with mean 2, $M_X(s) = \frac{0.5}{0.5-s}$, $\forall s < 0.5$.

(B) $E[X] = \frac{dM_X(s)}{ds}|_{s=0}$.

(C) $E[X^2] = -\frac{d^2 M_X(s)}{ds^2}|_{s=0}$.

(D) If X is a standard normal random variable, $M_X(s) = e^{s^2}$.

(E) If X is a normal random variable such that $\mathrm{E}[X] = 0$ and $\mathrm{E}[X^2] = 4$, $M_X(s) = e^{2s^2}$.

十八. Let X and Y be independent random variables that are uniformly distributed on the interval $[0, 1]$. Define $Z = \frac{X}{Y}$. Let f_Z be the probability density function of Z and F_Z be the cumulative distribution function of Z. Which of the following statements is/are true?

(A) $F_Z(\frac{1}{2}) = \frac{1}{4}$.

(B) $F_Z(2) = \frac{1}{2}$.

(C) $f_Z(\frac{1}{2}) = \frac{1}{4}$.

(D) $f_Z(2) = \frac{1}{8}$.

(E) $P(Z \geq 2) = P(Z \leq 0.5)$.

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十九. A defective coin minting machine produces coins whose probability of heads is a random variable X with CDF $F_X(x) = x^2$, $\forall x \in [0,1]$. A coin produced by this machine is selected and tossed twice, with successive tosses assumed independent. Let Ω be the sample space. Let Y_1 be a random variable that represents the number of heads in the first toss of the selected coin. Let Y_2 be a random variable that represents the number of heads in the first two tosses of the selected coin. Let A be the event that the first coin toss results in head. Namely, $A = \{Y_1 = 1\} = \{\omega \in \Omega | Y_1(\omega) = 1\}$. Let $f_{X|A}$ be the conditional PDF of X given event A. Which of the following statements is/are true?

(A) $P(Y_1 = 1) = \frac{1}{2}$.

(B) $P(Y_1 = 0) = \frac{1}{3}$.

(C) $f_{X|A}(\frac{1}{2}) = \frac{4}{5}$.

(D) $f_{X|A}(\frac{1}{3}) = \frac{1}{3}$.

(E) $P(Y_2 = 0) = \frac{1}{6}$.

二十. Which of the following statements is/are true?

(A) If X is an exponential random variable with mean 1, then $P(X \geq a) \leq \frac{1}{a}$, $\forall a > 0$.

(B) If X is a standard normal random variable, then $P(X^2 + 2X + 1 \geq a) \leq \frac{3}{a}$, $\forall a > 0$.

(C) If X is a normal random variable such that $\mathrm{E}[X] = 1$ and $\mathrm{E}[X^2] = 4$, then $P(|X - 1| \geq 2) \leq \frac{3}{4}$.

(D) If Y_1, Y_2,... are independent and identically distributed random variables with mean 5, then $\lim_{n\to\infty} P(|5 - \frac{1}{2n}\sum_{k=1}^{n} Y_k| \geq 0.01) = 0$.

(E) If X_1, X_2,... are independent Poisson random variables with variance 4 and $Y_n = \frac{\sqrt{n}}{2} \times (4 - \frac{1}{n}\sum_{k=1}^{n} X_k)$, then $\lim_{n\to\infty} P(Y_n \leq 1) \leq 0.5$.