台灣聯合大學系統105學年度碩士班招生考試試題

類組:電機類 科目:工程數學 B(3004)

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※請在答案卷內作答

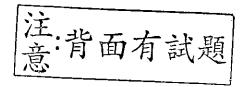
Note: Detailed derivations are required to obtain a full score for each problem.

1. (10%) Let
$$A = \begin{pmatrix} 0 & 3 & 2 & 1 & -4 \\ 2 & 10 & 10 & 16 & 14 \\ -3 & 0 & -5 & -2 & -7 \\ -2 & -1 & -4 & -3 & -6 \\ 2 & 7 & 8 & 11 & 10 \end{pmatrix}$$
 and $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

- (a) (3%) Compute rank(A).
- (b) (2%) Compute rank(AB).
- (c) (3%) Compute rank $(A^t A A A^t)$.
- (d) (2%) Compute $\dim(N(B^tA))$.
- 2. (10%) Let V be the vector space spanned by the ordered basis functions $\beta = \{xe^{ax}, e^{ax}, e^{bx}\}$ where $a, b \in R$ and $a \neq b$. Define a linear transformation $T : V \to V$ with parameters $p, q \in R$:

$$\mathsf{T}(y(x)) = y'' + py' + qy.$$

- (a) (4%) Find the matrix representation for $[T]_{\beta}$.
- (b) (6%) There are two conditions for p and q such that $\dim(N(T)) = 2$. For each condition, express p and q in terms of a and b, and also find the corresponding null space.
- 3. (5%) Let A and B be $n \times n$ square matrices such that AB = C where C is an upper triangular matrix with $C_{ij} \neq 0$ whenever $j \geq i$. Prove that A and B are both invertible.
- 4. (16%) Let V be a vector space over a field F, T be a linear operator on V, and W be a subspace of V. We say that W is invariant under T if for each vector v in W the vector Tv is also in W. Let W be an invariant subspace for T, and v ∈ V. The T-conductor of v into W, denoted by S_T(v, W), is defined as the set of all polynomials g(x) over F such that g(T)v is in W, i.e., S_T(v, W) = {g(x) ∈ F[x] | g(T)v ∈ W}.
 - (a) (8%) Prove the following statement. If W is an invariant subspace for T, then, for each polynomial $g(x) \in \mathbb{F}[x]$, W is invariant under g(T).
 - (b) (8%) Prove that if W is an invariant subspace for T then $S_T(v, W)$ is a subspace of $\mathbb{F}[x]$, the set of polynomials over \mathbb{F} .
- 5. (9%) Let T be a linear operator on a finite-dimensional inner product space V. Prove that $N(T^*T) = N(T)$, where N(T) is the null space for T.





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6. (10%)

- (a) (3%) Prove that for any two events A and B, we have $P(A \cap B) \ge P(A) + P(B) 1$.
- (b) (5%) Generalize to the case of n events A_1, A_2, \dots, A_n , by showing that $P(A_1 \cap A_2 \cap \dots \cap A_n) \geq P(A_1) + P(A_2) + \dots + P(A_n) (n-1)$.
- (c) (2%) Let A_1, A_2, \dots, A_n be n events. Show that if $P(A_1) = P(A_2) = \dots = P(A_n) = 1$, then $P(A_1 \cap A_2 \cap \dots \cap A_n) = 1$.
- 7. (10%) Let X_0 be the amount of rain that will fall in the United States on next Christmas day. For k > 0, let X_k be the amount of rain that will fall in the United States on Christmas k years later. Let N be the smallest number of years that elapse before we get a Christmas rainfall greater than X_0 . Suppose that $P(X_i = X_j) = 0$ if $i \neq j$; the events concerning the amount of rain on Christmas days of different years are all independent, and the X_k 's are identically distributed. Please show that the probability mass function of N is $P(N = n) = \frac{1}{n(n+1)}$, $n \geq 1$. (Hint: You can calculate P(N > n) first.)
- 8. (5%) Let θ be a random number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Find the probability density function of $X = \tan \theta$.
- 9. (5%) The joint PDF of X, Y is the following,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

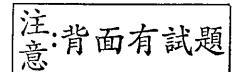
We know that X, Y are normal random variables. Find Var(X + Y).

10. (10%) Let X_1, X_2, \ldots, X_9 be independent Poisson random variables with the following PMF:

$$P_{X_i}(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$
, with $\lambda = 1$

(a) (5%) Use the Markov inequality to obtain a bound on

$$P\left\{\sum_{i=1}^{9} X_i \ge 15\right\}$$





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(b) (5%) Use the two-sided Chebyshev inequality to obtain a bound on

$$P\left\{\sum_{i=1}^{9} X_i \ge 15\right\}$$

11. (10%) Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \le 3\\ 0, & \text{otherwise} \end{cases}$$

and let A be the event $\{X \ge 2\}$.

- (a) (5%) Find E[X|A].
- (b) (5%) Let $Y = X^2$. Find E[Y] and var(Y).

