1.True or False? [21 points] Please provide one or two sentences to justify your answer.

(a) [3 points] If $ax \equiv ay \pmod{n}$, then $x \equiv y \pmod{n}$.

(b) [3 points] If $x \equiv y \pmod{n}$, then $ax \equiv ay \pmod{n}$.

(c) [3 points] The set of integers and the set of prime numbers have the same cardinality.

(d) [3 points] If a relation is symmetric and transitive, then the relation is reflexive.

(e) [3 points] $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$.

(f) [3 points] 5 divides $n^5 - n$ whenever n is a positive integer.

(g) [3 points] Given $a, b \in \mathbb{N}^+$ and $gcd(a, b) \neq 1$, we cannot find an inverse of a modulo b in some cases.

2. Independence is the key to a good research! [12 points] An independent set in a graph is defined as a set of vertices with no edges connecting them, i.e., no two of which are adjacent. Let G be a graph with $nd/2$ edges $(d > 1)$. Here, we conduct the following probabilistic experiment for finding an independent set in G: delete each vertex of G with all its incident edges independently with probability $1 - 1/d$.

(a) [6 points] Compute the expected number of vertices and edges that remain after the deletion process.

(b) [6 points] Based on (a), try to infer that for any graph with n vertices with $nd/2$ edges, there is an independent set with at least $n/2d$ vertices.

3. A tree is a seed that never gave up on its dream to flourish. [10 points] Let T be a spanning tree of a graph G with an edge cost function c. T is defined to have the **cycle property** if for any edge $e' \notin T$, $c(e') \geq c(e)$ for all e in the cycle generated by adding e' to T. Also, T is defined to have the **cut property** if for any edge $e \in T$, $c(e) \leq c(e')$ for all e' in the cut defined by e. Show that the following three statements are equivalent:

1. T has the cycle property.
2. T has the cut property.
3. T is a minimum cost spanning tree.

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4. Respect for the ancients. [8 points] Find an integer x such that $x \equiv 1 \pmod 3$, $x \equiv 3 \pmod 7$ and $x \equiv 9 \pmod{11}$.

5. Show time! [10 points] Prove that for every positive integer n, there are n consecutive composite integers. In other words, prove that we can find n consecutive composite integers for any n.

6. I want to play a game! [13 points] Now we want to play a famous game called "Sprouts", which is a two-player game and can be played with paper and pencil. First, several dots are drawn on the paper. Afterward, the players take turns, each doing the following process.

- Drawing a line that connects two dots or connects a dot to itself but does not touch or cross any other line.
- Putting a new dot on this new line, thus separating it into two lines.

If no dot can have more than three lines attached to it, the last player that can make a legal move wins.
(a) [6 points] Prove that any Sprouts game consists of a finite number of moves before someone loses. In other words, the game will terminate eventually.
(b) [7 points] Show a tight upper bound on the worst-case number of moves in a Sprouts game that starts with n dots, and prove your answer.

7. Act together, we go far. [12 points] Suppose we have two isomorphic graphs G_1 and H_1, as well as two isomorphic graphs G_2 and H_2. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are also isomorphic.

8. Simplicity is the keynote of all true elegance. [14 points]
(a) [7 points] Use the recurrence tree method to solve the time complexity of recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$ in the Θ-notation.
(b) [7 points] Solve the time complexity of recurrence $T(n) = 2T(\lfloor n/2 \rfloor + \sqrt{n}) + \sqrt{n/4}$ in the Θ-notation without using Master method.