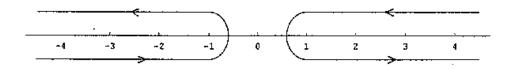1. (20 pts) Solve the general solution of the following coupled differential equations.

$$\frac{d^2}{dt^2}x_1(t) = x_1(t) - x_2(t)$$
$$\frac{d^2}{dt^2}x_2(t) = -x_1(t) + 3x_2(t)$$

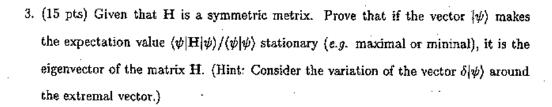2. (15 pts) Evaluate the series $\sum_{n=1}^{\infty}(-1)^n/n^4$.

   One way to calculate the sum is by contour integral technique, using the fact that $1/\sin(z\pi)$ has poles along the real axis at $z = 0, \pm 1, \pm 2, \ldots$. Consider the following contour:

   You can assume the residue theorem

$$\frac{2\pi i}{(n-1)!}f^{(n-1)}(z) = \int_C \frac{f(z')}{(z'-z)^n}dz'$$

   where $f(z)$ is regular at z.

3. (15 pts) Given that **H** is a symmetric metrix. Prove that if the vector $|\psi\rangle$ makes the expectation value $\langle\psi|\mathbf{H}|\psi\rangle/\langle\psi|\psi\rangle$ stationary (e.g. maximal or mininal), it is the eigenvector of the matrix **H**. (Hint: Consider the variation of the vector $\delta|\psi\rangle$ around the extremal vector.)

4. (25 pts) Use the Fourier transform method to solve the following equation.

$$(\nabla^2 - \kappa^2)\psi(x, y, z) = \delta(x)\delta(y)\delta(z)$$

   with the boundary condition $\psi \to 0$ when $\sqrt{x^2 + y^2 + z^2} \to \infty$, where κ is a positive constant.

5. (25 pts) A certain radioactive sample is expected to undergo 3 decays per minutes. A student measures the number n of decays in 100 seperate one-minute intervals.

   (a) How many times does he expect to see 1 decay in one minute? 5 decays in one minute?

   (b) What is the expected mean and the standard deviation of the distribution of n?

   (c) If the student measures the time between two consecutive decays, Δt. What is the expected Δt distribution?