所別：天文研究所碩士班 科目：應用數學

(1) (10 points)
x, y, z are three functions in which only two of them are independent (i.e., the third one can be expressed as a function of the other two, e.g., $x = x(y, z)$). Show that

$$\left(\frac{\partial x(y,z)}{\partial y}\right)_z \left(\frac{\partial y(z,x)}{\partial z}\right)_x \left(\frac{\partial z(x,y)}{\partial x}\right)_y = -1 .$$

(2) (10 points)
x and y are real, and $i = \sqrt{-1}$. Show that

$$\tan(x + i\,y) = \frac{\sin(2x) + i\,\sinh(2y)}{\cos(2x) + \cosh(2y)} .$$

(3) (20 points)
(a) (15 points) Show that in three-dimensional geometry $\Phi = 1/|\vec{r} - \vec{r}_s|$ satisfies the Poisson equation $\nabla^2\Phi = -4\pi\delta(\vec{r} - \vec{r}_s)$, where $\vec{r}_s$ is a constant, and δ is the Dirac delta-function. What are the corresponding Φ in two-dimensional and one-dimensional geometry?
(b) (5 points) Derive the solution $\psi(\vec{r})$ for the Poisson equation $\nabla^2\psi = 4\pi G\rho(\vec{r})$.

(4) (20 points)
(a) (5 points) Solve the Fredholm integral equation $y(x) = x^7 + \int_0^1 x^2 t^2 y(t)\,\mathrm{d}t$.
[Hint: $\int_0^1 t^2 y(t)\,\mathrm{d}t$ is a constant.]
(b) (15 points) Solve the Volterra integral equation $y(x) = x^7 + \int_0^x x^2 t^2 y(t)\,\mathrm{d}t$.
[Hint: Change it to an ordinary differential equation.]

(5) (20 points)
Solve the following set of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + 2y + z\,, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = x + z\,.$$

The initial conditions are $x = y = z = 1$ at $t = 0$.

(6) (20 points)
One-dimensional Navier-Stokes equation with negligible pressure can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} ,$$

where the velocity u is a function of t and x. Assuming constant viscosity ν, let

$$u = -2\nu\frac{\partial \log\phi}{\partial x}$$

then derive a partial differential equation for $\phi(t, x)$. From the general solution of this partial differential equation, write down the solution for $u(t, x)$.