國立中央大學八十八學年度碩士班研究生入學試題卷

所別: 天文研究所 不分組 科目: 應用數學 共 一 頁 第 一 頁

PLEASE READ THIS SHORT MESSAGE FIRST:
Please work out the following problems in detail, otherwise put down how you may proceed. Attempt as many problems as you can but spend your time *wisely*. Please pay attention to the score of each problem. Good luck!

(1) (25 points)
The relations between the spherical coordinates (r, θ, ϕ) and Cartesian coodinates (x, y, z) are $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

(a) Write down the Cartesian components of the unit vectors in the direction of increasing r, θ and ϕ (i.e., $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$).
(b) The position vector of a particle is given by $r\hat{e}_r$. Find the velocity and acceleration of the particle in spherical coordinates. Identify the centripetal acceleration and the Coriolis acceleration.
(c) Compute $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ in spherical coordinates.

(2) (25 points)
The adjoint of a matrix M is defined as the transpose of its complex conjugate and is denoted by $\mathbf{M}^\dagger$. A matrix U is unitary if its inverse is equal to its adjoint, i.e., $\mathbf{U}^{-1} = \mathbf{U}^\dagger$.

(a) If λ is an eigenvalue of a unitary matrix, show that $|\lambda| = 1$.
(b) Show that the eigenvectors corresponding to different eigenvalues of a unitary matrix are orthogonal.
(c) Verify (a) and (b) by finding the eigenvalues and eigenvectors of the following matrix,

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(3) (25 points)
The relation between the rate of change of a vector in an inertial frame and a rotating frame is

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{S}}{dt}\right)_{\text{rotating}} + \vec{\Omega} \times \vec{S},$$

where $\vec{\Omega}$ is the angular velocity of the rotating frame and is a constant. Solve $\vec{S}$ in the rotating frame if $\vec{\Omega} = \Omega\hat{e}_z$ and the rate of change in the inertial frame is a constant vector $\vec{A} = A\hat{e}_y$.
[Hint: Let $\vec{S} = \vec{S}' + (\vec{A} \times \vec{\Omega})/\Omega^2$.]

(4) (25 points)
Define Fourier transform and its inverse as:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt, \qquad f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega.$$

(a) Find the Fourier transform of $f(t)$, where $f(t) = 1$ if $|t| \le a$ and $f(t) = 0$ if $|t| > a$.
(b) Show the Parseval's relation: $\int_{-\infty}^{\infty} f(t)\bar{g}(t)\, dt = \int_{-\infty}^{\infty} F(\omega)\bar{G}(\omega)\, d\omega$, where the bar denotes complex conjugate.
(c) Hence evaluate the integral $\int_{-\infty}^{\infty} (\sin t/t)^2\, dt$.