

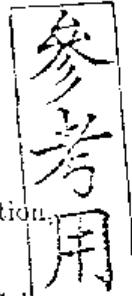
PLEASE READ THIS SHORT MESSAGE FIRST:

Please work out the following problems in detail, otherwise put down how you may proceed. Attempt as many problems as you can but spend your time wisely. Please pay attention to the score of each problem. Good luck!

- (1) (20 points) What is the name of the following equation?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

What is the meaning of a ? If a is a constant, show that $u = g(s)$ is a solution of the above equation, where $g(s)$ is an arbitrary analytic function of s , and s is given by $st - \sqrt{1-s^2}y = at$.



- (2) (20 points) What is a linear ordinary differential equation? Give one property of its solutions. Solve the following set of equations

$$\frac{dx}{dt} = x + 4y, \quad \frac{dy}{dt} = 2x + 3y,$$

subject to the conditions $x = x_0$, $y = y_0$ at $t = 0$.

- (3) (20 points) What is divergence theorem in the context of vector analysis? Show that

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}_*|} = -4\pi\delta(|\vec{r} - \vec{r}_*|),$$

where the differentiation is with respect to \vec{r} . [Hints: first show that the left hand side is zero if $\vec{r} \neq \vec{r}_*$. Then use divergence theorem to show that the integral of the left hand side over a volume including \vec{r} is equal to -4π .]

Consider a vector $\vec{V} = -\vec{\nabla}\phi + \vec{\nabla} \times \vec{A}$, where

$$\phi(\vec{r}) = \frac{1}{4\pi} \int \frac{s(\vec{r}_*)}{|\vec{r} - \vec{r}_*|} d^3 r_*, \quad \vec{A}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{\omega}(\vec{r}_*)}{|\vec{r} - \vec{r}_*|} d^3 r_*,$$

Show that $\vec{\nabla} \cdot \vec{V} = s$ and $\vec{\nabla} \times \vec{V} = \vec{\omega}$. [Hints: use the first result of this problem.]

Suppose $\vec{V} = \vec{\Omega} \times \vec{r}$. If $\vec{\Omega}$ is a constant, find $\vec{\omega}$. Can you find \vec{A} ? If yes, what is it? If no, why?
[Note: $\vec{\nabla} \times (\vec{P} \times \vec{Q}) = \vec{P}(\vec{\nabla} \cdot \vec{Q}) - \vec{Q}(\vec{\nabla} \cdot \vec{P}) + (\vec{Q} \cdot \vec{\nabla})\vec{P} - (\vec{P} \cdot \vec{\nabla})\vec{Q}$]

- (4) (20 points) $f(z) = u(x, y) + i v(x, y)$ is an analytic function of $z = x + iy$, where u , v , x and y are real and $i = \sqrt{-1}$. Write down the Cauchy-Riemann relation. If $z = r \exp(i\theta)$, write down the Cauchy-Riemann relation in polar coordinates (r, θ) . What is $\partial f / \partial \bar{z}$? Show that $\partial f / \partial \bar{z} = 0$ is equivalent to the Cauchy-Riemann relation ($\bar{z} = x - iy$ is the complex conjugate of z). What is a Laurent series? What is the relation between Laurent series and residues? Find the Laurent series of $g(z) = z^{-1}(1-z)^{-1}$ (please state the range of validity of each series). Write down the residue of $g(z)$ at each singular point.

- (5) (20 points) What is a Fourier series? Consider the one dimensional inhomogeneous diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + s(t, x),$$

where the source $s(t, x)$ is a given function. This equation is subject to initial condition $u(0, x) = f(x)$ when $t = 0$, and boundary conditions $u(t, 0) = u(t, L) = 0$ at $x = 0$ and L . Try the following Fourier series

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L}, \quad s(t, x) = \sum_{n=1}^{\infty} s_n(t) \sin \frac{n\pi x}{L}.$$

Find the ordinary differential equation governing $u_n(t)$. What is the corresponding initial or boundary condition on $u_n(t)$? If $s_n(t) = \exp(-t/\tau_n)$ and $f(x) = \sin(\pi x/L)$, write down the solution $u(t, x)$.