國立中央大學103學年度碩士班考試入學試題卷

所別：天文研究所碩士班 不分組(一般生)　科目：應用數學　共 2 頁　第 1 頁
天文研究所碩士班 不分組(在職生)

本科考試禁用計算器

＊請在試卷答案卷（卡）內作答

1. (total 15%) For the binomial distribution $p_b(x)=\dfrac{N!}{x!(N-x)!}p^x(1-p)^{N-x}$

   where $0\le p\le 1$, N and x are integers and $0\le x\le N$, calculate

   (i) (5%) Prove $\sum_{x=0}^{N} p_b(x)=1$

   (ii) (5%) calculate the mean: $\langle x\rangle \equiv \sum_{x=0}^{N} x\cdot p_b(x)$ and

   (iii) (5%) calculate the variance: $\sigma_x^2 \equiv \left\langle \left(x-\langle x\rangle\right)^2\right\rangle = \sum_{x=0}^{N}\left(x-\langle x\rangle\right)^2 p_b(x)$.

2. (total 10%) The Gamma function is defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

   (i) (5%) Prove that

$$\Gamma(x+1)=x\Gamma(x)$$

   (ii) (5%) Calculate $\Gamma\left(\dfrac{1}{2}\right)$

3. (14%) Suppose a vector $\vec{V}$ can be expressed in two different orthogonal coordinate systems as

$$\vec{V}=\sum_{i=1}^{3}V_i\hat{x}_i=\sum_{i=1}^{3}V_i'\hat{x}_i'$$

   where $\begin{cases}\hat{x}_i\cdot\hat{x}_j=\delta_{ij}\\ \hat{x}_i'\cdot\hat{x}_j'=\delta_{ij}\end{cases}$ and $\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$

   Show that the transformation between components (V_1,V_2,V_3) and (V_1',V_2',V_3') can be written in a matrix form as

$$\begin{pmatrix}V_1'\\V_2'\\V_3'\end{pmatrix}=\begin{pmatrix}\hat{x}_1'\cdot\hat{x}_1 & \hat{x}_1'\cdot\hat{x}_2 & \hat{x}_1'\cdot\hat{x}_3\\ \hat{x}_2'\cdot\hat{x}_1 & \hat{x}_2'\cdot\hat{x}_2 & \hat{x}_2'\cdot\hat{x}_3\\ \hat{x}_3'\cdot\hat{x}_1 & \hat{x}_3'\cdot\hat{x}_2 & \hat{x}_3'\cdot\hat{x}_3\end{pmatrix}\begin{pmatrix}V_1\\V_2\\V_{3'}\end{pmatrix}$$

4. (10%) The Fibonacci series, 1, 1, 2, 3, 5, 8, 13, 21 ……, obeys the recurrent relation as $a_{n+1} = a_n + a_{n-1}$ and $a_1 = a_2 = 1$, calculate

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

5. (total 10%) Find the general solutions of following equations

(i) (5%) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2$

(ii) (5%) $\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = x + y \end{cases}$

6. (total 16%) Any complex number z can be written as $z = re^{i\theta}$ where $r \ge 0$ and $0 \le \theta < 2\pi$, find r and θ for the following complex numbers

(i) (4%) -5 (ii) (4%) $\ln(-1)$ (iii) (4%) $(1+i)^i$ (iv) (4%) 2^{2+2i}

7. (total 10%)Prove that

(i) (5%) An $n \times n$ matrix $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{S} + \mathbf{A}$ where $\mathbf{S}$ is a symmetry matrix, that is $\mathbf{S}^{\mathrm{T}} = \mathbf{S}$, and $\mathbf{A}$ is a skew-symmetry matrix, that is $\mathbf{A}^{\mathrm{T}} = \mathbf{-A}$.

(ii) (5%) $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ where both $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices.

8. (total 15%)The Fourier transform of a function $f(t)$ is defined as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Show that

(i) (5%) If $f(t)$ is a real function, $F(-\omega) = [F(\omega)]^*$

(ii)(5%) The Fourier transform of $f(t+t_0)$ is $e^{-i\omega t_0}F(\omega)$ where t_0 is a constant

(iii) (5%) The Fourier transform of $f(at)$ is $\dfrac{1}{|a|}F\left(\dfrac{\omega}{|a|}\right)$ where a is a real constant and $a \ne 0$

參考用

注意：背面有試題