所別：統計研究所碩士班 一般生 科目：基礎數學
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In what follows, I_k denotes the $k \times k$ dimensional identity matrix, $\det(A)$ is the determinant of the square matrix A, and A' is the transpose of the matrix A.

1. a) Let P and Q be $n \times n$ dimensional non-singular matrices such that $Q = P + UV$, where U has dimension $n \times q$ and V has dimension $q \times n$. Show that

$$Q^{-1} = P^{-1} - P^{-1}U(I_q + VP^{-1}U)^{-1}VP^{-1}.$$

[10 %]

b) Let $X_n = (\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n)'$ be an $n \times k$ dimensional matrix where the $\underline{\mathbf{x}}_i$'s are k-vectors. Show that

$$(X'_{n-1}X_{n-1})^{-1} = (X'_nX_n)^{-1} + \frac{(X'_nX_n)^{-1}\underline{\mathbf{x}}_n\underline{\mathbf{x}}'_n(X'_nX_n)^{-1}}{1 - \underline{\mathbf{x}}'_n(X'_nX_n)^{-1}\underline{\mathbf{x}}_n}.$$

[5 %]

2. Find $\int_1^\infty \frac{|\sin x|}{x}\, dx.$ [10 %]

3. Let $f_n(x) = nx/(1 + n^2x^2), n = 1, 2, \ldots$. Prove or disprove

a) $\{f_n(x)\}_{n=1}^\infty$ is convergent. [5 %]

b) $\{f_n(x)\}_{n=1}^\infty$ is uniformly convergent. [5 %]

c) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx.$ [5 %]

4. Given $x_1, x_2, \ldots, x_n$, let

$$f(\mu, \sigma) = \left[\frac{1}{\sqrt{2\pi(\sigma+1)}}\right]^n \exp\left\{-\frac{1}{2}\sum_{i=1}^n (x_i - \mu)^2/(\sigma+1)\right\}, -\infty < \mu < \infty, \sigma > 0.$$

Find the values of μ and σ (in terms of x_i's) such that $f(\mu, \sigma)$ is maximized. [10 %]

5. Assume the following integrals exist. Prove that for any $f(x) > 0$ in D such that $\int_D f(x)dx = 1$,

$$\int_D \frac{g^2(x)}{f(x)}dx \geq (\int_D |g(x)|dx)^2.$$

Furthermore, find the $f(x)$ that attains the lower bound. [10 %]

6. Let X be an $n \times k$ dimensional matrix with $n > k$ and let $H = X(X'X)^{-1}X'$. Find the ranks of H and $I_n - H$, respectively. [10 %]

7. Show that $\det(I_p + AB) = \det(I_q + BA)$, where A and B are matrices of dimension $p \times q$ and $q \times p$, respectively. [10 %]

注意：背面有試題

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8. Let $\sum_i a_i$ and $\sum_i b_i$ be convergent sequences of positive numbers such that $\sum_i a_i \geq \sum_i b_i$. Show that

$$\sum_i a_i \log \frac{b_i}{a_i} \leq 0,$$

and the equality being attained when and only when $a_i = b_i$, for all i. (Note that $\log e = 1$.) [10 %]

9. Let A be an $m \times m$ symmetric matrix, $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_m$ be its eigenvalues, and let $\underline{\mathbf{x}}$ be an m-vector. Find

$$\sup_{\underline{\mathbf{x}}} \frac{\underline{\mathbf{x}}' A \underline{\mathbf{x}}}{\underline{\mathbf{x}}' \underline{\mathbf{x}}} \quad \text{and} \quad \inf_{\underline{\mathbf{x}}} \frac{\underline{\mathbf{x}}' A \underline{\mathbf{x}}}{\underline{\mathbf{x}}' \underline{\mathbf{x}}}.$$

[10 %]