國立中央大學九十三學年度碩士班研究生入學試題卷 共_一_頁 第_一_頁

所別：統計研究所碩士班 不分組 科目：基礎數學

1. Let $f(x)$ and $g(x)$ be continuous on $[a,b]$, $a,b \in R$, $a<b$ and differentiable on (a,b), such that $g'(x) \neq 0$, $\forall x \in (a,b)$. Show that there exists $c \in (a,b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(c)-f(a)}{g(b)-g(c)}$. (10%)

2. If $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$, and $\lim_{x\to a} g(x)(f(x)-1) = \alpha$. Find $\lim_{x\to a} f(x)^{g(x)}$. (10%)

3. Find $\displaystyle\lim_{n\to\infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$. (10%)

4. Find $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\,dx$, $(\log e = 1)$. (10%)

5. Let $F(x) = \begin{cases} 1-\dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, show that $F(x)$ is continuous and $F(x) \geq 0$ $\forall x \in R$. (10%)

6. Discuss the solutions of the system of equations

$$\begin{cases} x+y+z+t=4 \\ x+ay+z+t=4 \\ x+y+az+(3-a)t=6 \\ 2x+2y+2z+at=6 \end{cases} \quad \text{where } a \in R \text{ is fixed.}$$

(10%)

7. Let A be an $n \times n$ matrix. Show that

   (a) $(I_n - A)(I_n + A + A^2 + \cdots + A^k) = I_n - A^{k+1}$

   (b) If $A^\ell = 0$ for some $\ell \leq k \Longrightarrow I_n - A$ is invertible

   (c) Let $A = \begin{pmatrix} 2 & 2 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$, find A^{-1}. (10%)

8. Let $f \cdot R^3 \to R^2$ be defined by $f(x,y,z) = (2x-y, 2y-z)$. Determine the matrix of f relative to the ordered bases $\{(1,1,1),(0,1,1),(0,0,1)\}$ and $\{(0,1),(1,1)\}$. (10%)

9. Show that if $\det(A) \neq 0$, then $\det(\text{adj}A) = (\det A)^{n-1}$. ($\text{adj}A =$ the adjoint of A, $(\text{adj}A)_{ij} = (-1)^{i+j}\det(A_{ji})$) (10%)

10. Show that the eigenvectors corresponding to distinct eigenvalues are linearly independent. (10%)