系所別：統計研究所　科目：基礎數學

參考用

(1) For some real value p, $0 < p < 1$, define the function f by

$$f(x)=\begin{cases}(1-p)^{x-1}p, & x=1,2,3,\ldots\\ 0, & \text{otherwise}\end{cases}$$

(a) Calculate $F(y)=\sum_{x=1}^{[y]} f(x)$ for a given real value y, where $[y]$ denotes the largest integer $\le y$. **(6%)**

(b) Which of the following are true ? (multiple choice)

(i) $F(y)$ is differentiable　(ii) $F(y)$ is left-continuous

(iii) $F(y)$ is right-continuous　(iv) $F(y)$ is a step function

(v) $F(y)$ is increasing　(vi) $\lim_{y\to\infty} F(y)=1$ **(6%)**

(2) Define a real-valued function f by

$$f(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$$

(a) Show that for any $\alpha > 1$, $f(\alpha)=(\alpha-1)f(\alpha-1)$. **(6%)**

(b) Calculate $f(n)$ for any positive integer $n > 1$. **(6%)**

(c) The Stirling's formula states that for $n \in N$, $\lim_{n\to\infty}\frac{n!}{n^{n+(1/2)}e^{-n}}=\sqrt{2\pi}$.

Consider a sequence $A_n=\binom{2n}{n}p^n(1-p)^n$, $0<p<1$.

Show that $A_n \approx \frac{[4p(1-p)]^n}{\sqrt{n\pi}}$ as $n\to\infty$. **(7%)**

(d) What is the condition of p such that $\sum_{n=1}^{\infty} A_n$ will converge ? **(7%)**

Note: $\binom{m}{n}$ denotes possible number of combinations that we choose n items from m items.

(3) A function $f:(a,b)\mapsto R$ is *convex* on (a, b) if

$f(rx+(1-r)y)\le rf(x)+(1-r)f(y)$ for all $a<x<y<b$ and $0\le\lambda\le 1$.

(a) Which of the following are convex on $(0,\infty)$? (multiple choice)

(i) $1/x$ (ii) $\log x$ (iii) $-\log x$ (iv) e^{-x} (v) $e^{-(x-1)^2/2}$ (vi) $\tan^{-1}x$ **(6%)**

(b) An alternative definition for a convex function is that for $\mu\in(a,b)$,

$$f'(\mu)(x-\mu)+f(\mu)\le f(x) \quad \text{for all } x\in(a,b)$$

Show that if $f'(x)$ exists and $f''(x)\ge 0$ for all $x\in(a,b)$, then f is convex on (a, b). **(6%)**

(4) Let X be an n×3 matrix and $R=X'X=\begin{bmatrix}1&k&s\\k&1&t\\s&t&1\end{bmatrix}$.

(a) What is the constraint on k, s, t such that matrix R is *singular*. **(6%)**

(b) Suppose matrix R has three distinct eigenvalues $\lambda_1,\lambda_2,\lambda_3$ and let $\vec{v}_1,\vec{v}_2,\vec{v}_3$ be eigenvectors corresponding to $\lambda_1,\lambda_2,\lambda_3$. Please use matrix forms to represent the relationship between $\lambda_1,\lambda_2,\lambda_3,\vec{v}_1,\vec{v}_2,\vec{v}_3$ and R. **(6%)**

注意：背面有試題

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(c) Let the eigenvector matrix $B=(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ and suppose B is *orthonormal* (that is, $B'B=I_3$). The eigen decomposition of R is shown as

$$R=B\Lambda B'=(\vec{v}_1,\vec{v}_2,\vec{v}_3)\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}(\vec{v}_1,\vec{v}_2,\vec{v}_3)'$$

Calculate $\lambda_1+\lambda_2+\lambda_3$. **(6%)**

(d) If X represents a data matrix, then $Y=XB$ is commonly used for the dimension reduction purpose. Show that $trace(Y'Y)=3$. **(6%)**

(5) Let A be an $n\times n$ matrix with the dimension of the row space is k, $k<n$.

(a) Which of the following are true ? (multiple choice)

(i) dimension of column space is k (ii) A is invertible

(iii) $\det(A'A)=0$ (iv) $\det(A^2)=0$ (v) all eigenvalues of A are zero

(vi) all row vectors are linearly independent **(6%)**

(b) Let $A=\begin{bmatrix}0&0&4&1\\0&2&0&4\\0&1&0&2\\5&0&0&1\end{bmatrix}$, find an *orthonormal* basis for the column space. **(6%)**

(c) Suppose $\vec{v}_1,\vec{v}_2,\vec{v}_3$ is an orthogonal basis for the row space of A.

Denote $\|\vec{v}_1\|_{23}$ to be the length of vector $\vec{v}_1$ projected onto the vector $3\vec{v}_2+4\vec{v}_3$.

Calculate $\|\vec{v}_1\|_{23}$. **(6%)**

(6) A symmetric matrix A is said to be *positive definite* if $x'Ax>0$ for all nonzero vectors x. For what range of the number b is the following matrix positive definite ?

$$A=\begin{bmatrix}2&2&4\\2&b&8\\4&8&7\end{bmatrix}$$ **(8%)**