國立中央大學107學年度碩士班考試入學試題

所別： 統計研究所 碩士班 不分組(一般生)　　　　共2頁 第1頁

統計研究所 碩士班 不分組(在職生)

科目： 基礎數學

本科考試可使用計算器，廠牌、功能不拘　　　　*請在答案卷(卡)內作答

1. (20 points)

(a) (10 points) *Young's inequality*: Let $a > 0$ and $b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(Hint: Use the convex function $f(x) = e^x$ and choose $a^p = e^x$ and $b^q = e^y$.)

(b) (10 points) Suppose that

$$\int_0^\infty f(x)dx = 1 \quad \text{and} \quad \int_0^\infty e^{kx} f(x)dx < \infty$$

for $k > 0$. Prove that

$$\sqrt[t]{\int_0^\infty x^t f(x)dx}$$

is non-decreasing in t.

(Hint: Use *Young's inequality*.)

2. (20 points)

Let $f : [0, \infty) \mapsto [0, \infty)$ be a non-increasing function with $f(x) \to 0$, as $x \to \infty$. Then, there exists a continuous function $g : [0, \infty) \mapsto (0, \infty)$ such that

$$\int_0^\infty g(x)dx = +\infty \quad \text{but} \quad \int_0^\infty f(x)g(x)dx < \infty. \tag{1}$$

Let $\widetilde{f}$ be a strictly positive continuously differentiable with $\widetilde{f}(x) \geq f(x)$, for all $x \geq 0$, with $\widetilde{f}(x) \to 0$, as $x \to \infty$. With such $\widetilde{f}$ and $g = -\widetilde{f}'/\widetilde{f}$, prove that (1) holds.

注意:背面有試題

參考用

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3. (20 points)
(a) (6 points) Compute

$$\int_{-\infty}^{0}\int_{-y}^{\infty} e^{-(x^2+y^2)}dxdy.$$

(b) (7 points) Compute

$$\int\int\int_D e^{-(x^2+y^2+z^2)}dxdydz,$$

where $D = \{(x,y,z) : x^2+y^2 \le z^2,\ x \ge 0\}$.
(c) (7 points) Compute

$$\int_0^{\pi} \cos mx \cos nx dx,$$

for non-negative integers m, n.
(Need to show the details of the calculation)

4. (20 points)
A curve is given by

$$3x^2+5xy+3y^2 = 1.$$

Find the minimum distance from the points on the curve to the origin.

5. (20 points)
A $n \times n$ matrix P is idempotent and symmetric. Prove that
(a) (5 points) $I_n - P$ is also an idempotent matrix, where I_n is the $n \times n$ identity matrix.
(b) (5 points) P is a positive semi-definite matrix.
(c) (5 points) If $\text{rank}[P] = r \le n$, then it has r eigenvalues equal to unity and $n-r$ eigenvalues equal to zero.
(d) (5 points) $\text{tr}[P] = \text{rank}[P]$.

注意:背面有試題

參考用