科目：線性代數　　校系所組：中大數學系甲組　交大應用數學系甲組、乙組
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## LINEAR ALGEBRA 2007

1. (10 %) Find all points $(a, b, c)$ in $\mathbb{R}^3$ for which the system

$$\begin{cases} 2x + 4y + 6z = a \\ 4x + 5y + 6z = b \\ 7x + 8y + 9z = c \end{cases}$$

has at least one solution.

2. (10 %) Let $A, B$ be $n \times n$ matrices over $\mathbb{R}$. Prove or disprove the following statements.
   (1) The trace of $AB - BA$ is always zero.
   (2) If $AB = -BA$, then at least one of $A, B$ is not invertible.

3. (15 %) Let $A := \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $I$ be the $6 \times 6$ identity matrix.
   Fill the following blanks:
   (1) The eigenvalues of $A$ are ________
   (2) The determinant of $I + A$ is ________

4. (20 %) Let $V$ be a finite-dimensional vector space over a field $F$, and let $V^*$ denote the *dual space* of $V$ i.e., $V^*$ is the space of linear functions $V \to F$.
   (1) Show that $V^*$, as a vector space, is isomorphic to $V$.
   (2) For $v \in V$, define $\phi_v\colon V^* \to F$ by $\phi_v(f) := f(v)$ for $f \in V^*$. Show that the map $\phi\colon V \to (V^*)^*$ define by $v \mapsto \phi_v$ is an isomorphism of vector spaces.

5. (10 %) Let $V$ be a finite-dimensional inner product space over $\mathbb{R}$ with the inner product $\langle , \rangle\colon V \times V \to \mathbb{R}$. If $W$ is a subspace of $V$, we define

$$W^{\perp} := \{v \in V \mid \langle w, v\rangle = 0 \text{ for all } w \in W\}.$$

   Suppose $W_1, W_2$ are two subspaces of $V$. Show that

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}.$$

6. (20 %) For all $x \in \mathbb{R}^n$, we define the norm of $x$ by $\|x\| = \sqrt{\langle x, x\rangle}$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.
   Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
   (1) Find a vector $p$ in the column space of $A$ (the subspace of $\mathbb{R}^3$ spanned by the column vectors of $A$) such that $\|p - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{R}^4$
   (2) Find a vector $x_0$ in the row space of $A$ (the subspace of $\mathbb{R}^4$ spanned by the row vectors of $A$) such that $A \cdot x_0 = p$.

7. (15 %) Determine whether $A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$ are similar over $\mathbb{C}$ or not? Justify your answer.