國立中央大學九十一學年度碩士班研究生入學試題卷

所別: 數學系 不分組 科目: 數值分析 共 1 頁 第 1 頁

1. **(20 分: Fixed-point iteration)**
   Let $F \in C[a,b]$ be such that $F([a,b]) \subseteq [a,b]$. Suppose that F' exists on $[a,b]$ and a positive constant $k < 1$ exists with $|F'(x)| \le k$ for all $x \in [a,b]$. Then F has a unique fixed point p in $[a,b]$.

   (a) Show that for any $p_0 \in [a,b]$, the sequence $\{p_n\}$ defined by $p_n = F(p_{n-1})$, $n \ge 1$, converges to the unique fixed point p of F in $[a,b]$.

   (b) Suppose, in addition, that F' is continuous on $[a,b]$ and $F'(p) \neq 0$. Show that $\{p_n\}$ converges linearly to p.

2. **(20 分: Divided differences)**
   The following data are given for a polynomial $P(x)$ of unknown degree:

x	0	1	2
$P(x)$	2	-1	4

   Determine $P(x)$ if all third-order forward differences are 1. (<u>Hint</u>: The Newton forward-difference formula for function $f(x)$ is $P_n(x) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f(x_0)$, where $P_n(x)$ is the nth Lagrange polynomial and $x = x_0 + sh$.)

3. **(10 分: Numerical differentiation)**
   Assume that $f \in C^4[x_0 - h, x_0 + h]$. Using Taylor's theorem to derive the following three-point formula for approximating $f''(x_0)$:

$$f''(x_0) = \frac{1}{h^2}\Big\{ f(x_0 - h) - 2f(x_0) + f(x_0 + h) \Big\} - \frac{h^2}{12} f^{(4)}(\xi), \quad \text{for some } \xi \in (x_0 - h, x_0 + h).$$

4. **(10 分: Gaussian quadrature)**
   Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf'(-1) + cf(1) + df'(1)$$

   that has degree of accuracy (degree of precision) 3.

5. **(20 分: Error estimates for the approximate solution $\tilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$)**
   Let $A \in \mathbf{R}^{n \times n}$ be a given symmetric and positive definite matrix, and $\mathbf{b} \in \mathbf{R}^n$ be a given vector. Let $0 < \lambda_{\min} \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_{\max}$ be the n real eigenvalues of matrix A.

   (a) Show that the condition number $\mathcal{K}(A)$ of the nonsingular matrix A relative to the norm $\|\cdot\|_2$ is

$$\mathcal{K}(A) = \frac{\lambda_{\max}}{\lambda_{\min}}.$$

   (b) Suppose that $\tilde{\mathbf{x}}$ is an approximation to the solution of $A\mathbf{x} = \mathbf{b}$. Show that

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \le \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) \frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}, \quad \text{provide } \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{b} \neq \mathbf{0},$$

   where $\mathbf{r}$ is the residual vector for $\tilde{\mathbf{x}}$.

6. **(20 分: Iterative technique for solving $A\mathbf{x} = \mathbf{b}$)**
   Let $A = [a_{ij}] \in \mathbf{R}^{n \times n}$, $a_{ii} \neq 0$, for $i = 1, 2, \cdots, n$, and $\mathbf{b} \in \mathbf{R}^n$. The basic concept of iterative methods for solving the system of linear equations $A\mathbf{x} = \mathbf{b}$ is to convert the system into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ and lead to the iterations

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for } k = 1, 2, \cdots.$$

   (a) Let $A = D - L - U$, where D is the diagonal matrix of A, $-L$ is the strictly lower-triangular part of A, and $-U$ is the strictly upper-triangular part of A. Find T and $\mathbf{c}$ for the SOR method.

   (b) Assume, in addition, that A is a symmetric and positive definite matrix. Please give a necessary and sufficient condition on the parameter ω in the SOR method to ensure the convergence of the method for any initial guess $\mathbf{x}^{(0)} \in \mathbf{R}^n$.