1. Solve the following initial value problems and find the maximum interval of existence of each solution (20 points).

   (a) $y' = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$.

   (b) $xy' + 2y = 4x^2, y(1) = 2$.

2. Using Laplace transform to solve the following system of linear differential equations (15 points)

$$\begin{cases} \dfrac{dx}{dt} = -x(t) + y(t) + e^t \\ \dfrac{dy}{dt} = -3x(t) + 2y(t) + 2e^t. \end{cases}$$

   satisfying the initial condition $x(0) = y(0) = 1$.

3. Solve

$$y(t) + 2\int_0^t cos(t-s)y(s)ds = 1. \ (10 points)$$

4. Find a pair of linearly independent solutions on an interval $0 < x < L$ of the differential equation (15 points)

$$xy'' + (1-x)y' - y = 0.$$

5. Find a general solution of the following differential equations (20 points)

   (a) $y'' + 4y = 3cscx$.

   (b) $x^2y'' - xy' + y = sin(lnx), x > 0$.

6. Let $\phi_1(t)$ and $\phi_2(t)$ be any two solutions of the following 2-dimensional linear system

$$\begin{cases} \dfrac{dx_1}{dt} = a(t)x_1 + b(t)x_2 \\ \dfrac{dx_2}{dt} = c(t)x_1 + d(t)x_2, \end{cases}$$

   where a, b, c, d are continuous functions on an interval $\mathbb{I}$. If t_0 is in $\mathbb{I}$, show that the Wronskian of $\phi_1(t), \phi_2(t)$ is given by

$$W(t) = W(t_0)exp\Big[\int_{t_0}^t (a(s) + d(s))ds\Big],$$

   where $t \in \mathbb{I}$. (20 points)