Select five problems out of six. Each problem is 20 points.

1. (a) (10 pts) Write down the formula of Newton's method for the solution of

$$\begin{cases} g_1(x_1, x_2) = x_1 + \sin x_2 = 0 \\ g_2(x_1, x_2) = x_1 \cos x_2 + x_2 = 0 \end{cases}$$

   (b) (10 pts) What causes the divergence of Newton's method ? How to prevent it ? (give some specific advice)

2. Many simple iterative methods transform $Ax = b$ into an equivalent form as $Mx = (M - A)x + b$, and then compute the approximate solutions by iterating $Mx_{k+1} = (M - A)x_k + b$.

   (a) (5 pts) What are major considerations to choose M ?

   (b) (15 pts) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, What are the M corresponding to Jacobi's method, Gauss–Seidel method, and SOR method with relaxation factor ω ?

3. (20 pts) Find a quadrature formula

$$\int_{-1}^{1} f(x)\, dx \approx c \sum_{i=0}^{2} f(x_i)$$

   that is exact for all quadratic polynomials

4. (20 pts) If a matrix A has absolute row sum less than 1,

$$|a_{i1}| + \cdots + |a_{ii}| + \cdots + |a_{in}| < 1 \quad \text{for each } i$$

   show from Gershgorin's theorem that all eigenvalues satisfy $|\lambda| < 1$.

5. (5 pts each) A is a matrix, prove that

   (a) A^T has the same eigenvalues as A.

   (b) $A^T A$ is a symmetric positive definite matrix.

   (c) if $A^T = -A$ then $x^T A x = 0 \ \forall x$.

   (d) if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ show that both eigenvalues are real.