國立中央大學八十五學年度碩士班研究生入學試題卷

所別: 數學研究所　不分組　　科目: 高等微積分　　共 1 頁 第 1 頁

每小題10分, 如作超過十小題時, 將只取得分較高的十題計分.

1. (a) Suppose $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove that $\sum_{k=1}^{\infty} |a_k|^p$ converges for $p \geq 1$.

   (b) Suppose $\sum_{k=1}^{\infty} a_k$ converges conditionally. Prove that $\sum_{k=1}^{\infty} k^p a_k$ diverges for all $p > 1$.

2. Suppose $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ and $f_n \to f$ uniformly on $[a, b]$.

   (a) Prove that $\{f_n\}$ is uniformly bounded and f is uniformly continuous on $[a, b]$.

   (b) Prove that $n^{-1}(f_1 + f_2 + \cdots + f_n)$ converges to f uniformly on $[a, b]$ as $n \to \infty$.

3. State the Mean Value Theorems for differentiation and integral, and prove one of them.

4. Suppose f is a real-valued differentiable function on $[a, b]$. Prove:

   (a) If $f'(a) < \lambda < f'(b)$, then $\lambda = f'(z)$ for some z in (a, b).

   (b) If f' is monotonic, then f' is continuous on $[a, b]$.

5. State the Weierstrass Approximation Theorem, and use it to prove that if f is a continuous function on $[a.b]$ such that $\int_a^b x^n f(x)dx = 0$ for all $n = 0, 1, 2, \cdots$, then $f \equiv 0$.

6. Let $f(0,0) = 0$ and $f(x, y) = \frac{x^3}{x^2+y^2}$ for $(x, y) \neq (0,0)$.

   (a) Does the directional derivative $D_u f(0,0)$ exist for any unit vector $u = (u_1, u_2)$?

   (b) Is f continuous at $(0,0)$? Why?

   (c) Is f differentiable at $(0,0)$? Why?

7. Is every continuous function on $[0, 1]$ of bounded variation ? If yes, prove it; if not, give a counterexample.

8. Let $f : [a, b] \to [m, M]$ be Stieltjes integrable with respect to a function α, and let $\phi : [m, M] \to R$ be a continuous function. Prove that the composite function $g = \phi \circ f$ is also Stieltjes integrable with respect to α.

9. Compute the explicit value of the integral $\int_0^{\infty} e^{-x^2} \cos(xt)dx$ for $-\infty < t < \infty$.