國立中央大學110學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)　　　　　　　　　　共2頁　第1頁
數學系 碩士班 應用數學組(一般生)
數學系 碩士班 應用數學組(在職生)

科目： 線性代數

本科考試禁用計算器　　　　　　　　　　　　*請在答案卷(卡)內作答

In this exam, V is assumed to be a vector space over some field $\mathbb{F}$, which will be explicitly indicated in the problems when necessary. Also recall that the set of all linear maps from V to $\mathbb{F}$ also forms a vector space over the same field $\mathbb{F}$ with the usual function addition and scalar multiplication. This space is called the *dual space of* V and is denoted by V^*. In set notation,

$$V^* = \{f : V \to \mathbb{F} \mid f \text{ is linear}\}$$

計算與證明題：請保留計算過程於答案紙上，無計算過程者不予計分。

1. Consider the following system of linear equations.

$$\begin{cases} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 = 7 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 = 7 \end{cases}$$

a. (5pts) Find the reduced row echelon form of its augmented matrix $(A|b)$.

b. (10pts) Explicitly find the solution set K of this linear system, including a particular solution and a basis for its homogeneous solution set K_H.

2. (15pts) Consider the following matrix

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}$$

Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$. If it is impossible, give a convincing reason.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$T(a, b, c, d) = (a + 2d, b - 2c, -2b + c, 2a + d)$$

Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the standard ordered basis for $\mathbb{R}^4$ (for example, $e_1 = (1, 0, 0, 0)$).

a. (5pts) Find the characteristic polynomial of T.

b. (5pts) Find the minimal polynomial $m(x)$ of T. Explain why your answer is indeed minimal.
Hint: by definition, $m(x)$ is the monic non-zero polynomial with smallest degree such that $m(T)$ is the zero map on $\mathbb{R}^4$.

注意:背面有試題

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4. (10pts) Let $V = \mathbb{C}^3$ be equipped with the standard inner product $<,>$. Let $f : V \to \mathbb{C}$ be the unique linear map defined by

$$f(a,b,c) = ia + (2+3i)b + (1-2i)c \qquad \forall (a,b,c) \in V.$$

In other words, f is an element in V^*. Find the unique vector $z \in V$ such that

$$f(v) = <v, z> \text{ for all } v \in V$$

5. (15pts) Let W be the subspace of $\mathbb{R}^4$ spanned by the following vectors

$$v_1 = (1,-1,0,0) \qquad v_2 = (0,1,-1,0) \qquad v_3 = (0,0,1,-1)$$

Apply the Gram-Schmidt algorithm to obtained an orthogonal basis for W.

6. (15pts) Let V be a finite dimensional vector space over $\mathbb{C}$. Let $T : V \to V$ be a linear map and let W be an T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_n$ are eigenvectors in V of T corresponding to **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $\mathbb{C}$. Prove that if $v_1 + v_2 + \cdots + v_n \in W$, then $v_i \in W$ for all $1 \le i \le n$. (*Hint: induction on n*)

7. (10pts) Let $V = \mathbb{R}^n$ be equipped with the standard inner product $<,>$. Suppose that $S = \{v_1, \ldots, v_m\}$ is a set of non-zero orthogonal vectors in V $(m \le n)$. Show that S is linearly independent.

8. (10pts) Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear map. Recall that there exists a unique linear map $T^t : W^* \to V^*$, called *the transpose of* T, where V^* and W^* are the dual spaces of V and W, respectively.

   Consider $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let $T : V \to W$ be the linear map defined by

$$T(a,b) = (3a+b, a-2b, 2a-b) \quad \forall a,b,c \in \mathbb{R}.$$

   Then there exists a unique linear map $T^t : W^* \to V^*$ as above. Define $\theta \in W^*$ by

$$\theta(x,y,z) = 3x - 2y + z \quad \forall x,y,z \in \mathbb{R}.$$

   Compute $T^t(\theta) \in V^*$.
   Hint: you need to either explicitly write down $T^t(\theta)(a,b)$ for any $(a,b) \in V$, or simply write down a 2×1 matrix representing $T^t(\theta)$.

注意:背面有試題