國立中央大學112學年度碩士班考試入學試題

所別： 工業管理研究所碩士班

科目： 作業研究

1. Consider a queueing model for which there is unlimited service, i.e., an infinite number of servers available. A self-service type situation is one example of such a model. Interarrival time is exponentially distributed with rate λ and service time is exponentially distributed with rate μ. Let the state be the number of customers in the system. One could use the general birth and death processes to analyze this model. Let the state be the number of customers in the system.

(a) (5 points) Draw a rate transition diagram for this model and determine λ_n and μ_n for state n

(b) (10 points) Let p_n be the steady-state probability of n customers in the system. Show that

$p_n = \frac{\lambda^n}{n!\mu^n} p_0, n = 1,2,\cdots$

(c) (10 points) Show that $p_n = \frac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n!}, n = 0, 1, \cdots$

2. Let $\mathrm{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$ be a transition probability matrix of a discrete time Markov chain. The eigenvalues of $\mathbf{P}$ are $\lambda_0 = 1, \lambda_1 = 1/6$. Apply Sylvester's formula to get the followings.

(a) (10 points) Determine $\mathbf{Z}_0$ and $\mathbf{Z}_1$

(b) (15 points) Determine the analytical solution of $\mathbf{P}^n$

Hint: (Sylvester's formula) Let $\mathbf{P}$ be a square matrix of order $s + 1$. Let $\lambda_0, \lambda_1, \cdots, \lambda_s$ be the eigenvalues of $\mathbf{P}$, and assume they are distinct. Then

$$\mathbf{P}^n = \sum_{k=0}^{s} \lambda_k^n \mathbf{Z}_k,$$

where matrix $\mathbf{Z}_k$ is given by

$$\mathbf{Z}_k = \frac{\prod_{r \neq k}(\mathbf{P} - \lambda_r \mathbf{I}_{s+1})}{\prod_{r \neq k}(\lambda_k - \lambda_s)},$$

$\mathbf{I}_{s+1}$ is the identity matrix of order $s + 1$.

3. (問答題，15 分) Consider the following linear programming problem (*P*) where **A** is the coefficient matrix, **x** is the decision variable vector, and **c** is the cost vector. The dual problem of (*P*), which is denoted as (*D*), is also formulated in the following.

$$(P): \text{Minimize } \mathbf{cx} \qquad (D): \text{Maximize } \mathbf{wb}$$
$$\text{subject to } \mathbf{Ax} \geq \mathbf{b} \qquad \text{subject to } \mathbf{wA} \leq \mathbf{c}$$
$$\mathbf{x} \geq \mathbf{0} \qquad \mathbf{w} \geq \mathbf{0}$$

Suppose that $\mathbf{x}_0$ is a feasible solution for (*P*), that is, $\mathbf{Ax}_0 \geq \mathbf{b}$, and $\mathbf{w}_0$ is a feasible solution for (*D*), that is, $\mathbf{w}_0\mathbf{A} \leq \mathbf{c}$. Please compare $\mathbf{w}_0\mathbf{b}$ (that is, the objective function value of (*D*)) and $\mathbf{cx}_0$ (that is, the objective function value of (*P*)) and determine: Do they have the same value? Or, is one larger than or equal to the other? Or, is one strictly larger than the other? You must provide convincing reasons to back up your answer (答案必須有明確的理由).

4. (計算題，15 分) Please solve the following linear programming problem and report the optimal solution. Show your calculation process in detail (答案必須有詳細的計算過程).

$$\text{Minimize } 6x_1 + 9x_2 + 15x_3 + 6x_4 + 9x_5$$
$$\text{subject to } 4x_1 - 4x_2 + 6x_3 + 2x_4 + 2x_5 >= 6$$
$$3x_1 + 3x_2 + 6x_3 + 3x_4 + 9x_5 >= 12$$
$$x_1, x_2, x_3, x_4, x_5 >= 0$$

5. (建立模型，20 分) John is a manager in charge of running a department of 20 Python programmers. Currently, he has a total of 15 projects that he can choose for his programmers. After some thinking, he came up with the following "incidence matrix" indicating the best composition for each project team. For example, John considers that project 1 is best to be handled by programmers 1, 3, …, and 20.

programmer	project 1	project 2	project 3	…	project 15
1	1	0	1	…	0
2	0	0	1	…	1
3	1	0	0	…	0
…	…	…	…	…	…
20	1	1	0	…	1

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As required by company policies, John must ensure that "no one in his department is left behind;" that is, every programmer must have at least one project to work on. Because the company must pay certain money to programmers doing each project, naturally they asked John to choose some of the projects so that as little money is paid as possible (while at the same time ensuring that no one is left behind).

Let $\mathbf{a}_{ij}$ denote the numeric value indicating whether programmer i can work on project j or not. For example, for project 1, we have $\mathbf{a}_{1,1}=1$, $\mathbf{a}_{2,1}=0$, $\mathbf{a}_{3,1}=1$, …, and $\mathbf{a}_{20,1}=1$). Let $\mathbf{c}_j$ be the total costs that the company will have to pay to those programmers doing project j. Moreover, let $\mathbf{X}_j$ be the decision as to whether John should choose project j or not. To be precise, $\mathbf{X}_j$ is a binary decision variable such that "$\mathbf{X}_j=1$" indicates that John should choose project j and "$\mathbf{X}_j=0$" indicates otherwise.

Please formulate the above situation as a binary linear programming model. Note: you are NOT allowed to define any more decision variable, parameters, symbols, etc. (不可自行定義任何新的決策變數、參數、或符號).