1. Suppose X_e and Y_e are independently distributed according to a Laplace distribution with pdf,

$$f_{X_e}(x) = f_{Y_e}(x) = \frac{\lambda}{2}e^{-\lambda|x|}, -\infty < x < \infty, \lambda > 0.$$

Consider the following random variable $R = |X_e| + |Y_e|$.
(a) (8 points) Show

$$f_R(r) = r\lambda^2 e^{-\lambda r}, r \geq 0.$$

(b) (7 points) Show

$$E[R] = \frac{2}{\lambda}$$

2. Suppose that X is an sample from Poisson(Θ) distribution, where Θ is Gamma distributed with parameter α and β,

$$f_\Theta(\theta) = \frac{\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, \theta > 0.$$

(a) What is the conditional probability of $X = x$ given $\Theta = \theta$? (5 points)
(b) What is the conditional probability density of $\Theta = \theta$ given $X = x$? (10 points)
(c) What is $E[\Theta \,|X = x]$? (5 points)

3. Let X be a nonnegative integer-valued random variable whose probability generating function $\Phi_X(t) = E[t^X]$ is finite for all t and let x_0 be a positive number. We have the following inequalities:

$$P(X \leq x_0) \leq \frac{\Phi_X(t)}{t^{x_0}}, 0 \leq t \leq 1;$$

$$P(X \geq x_0) \geq \frac{\Phi_X(t)}{t^{x_0}}, t \geq 1.$$

Suppose X is Poisson distributed with parameter λ. Verify

(a) $P(X \leq \lambda/2) \leq \left(\frac{2}{e}\right)^{\lambda/2}$ (8 points)

(b) $P(X \geq 2\lambda) \geq \left(\frac{e}{4}\right)^\lambda$ (7 points)

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4. Suppose the multivariate distribution of X_1 and X_2 is _uniform_ over the region $\{0 \le x_1 \le 1, x_1 \le x_2 \le (x_1+1)\}$, i.e. the joint probability density function is as following,

$$f(x_1, x_2) = \frac{1}{k} \quad \text{if } 0 \le x_1 \le 1 \text{ and } x_1 \le x_2 \le (x_1+1), \text{ or}$$

$$= 0 \quad \text{o.w.}$$

   (a) What is the value of k? (2pts)
   (b) Show the marginal distributions of X_1 and X_2, respectively. (8 pts)
   (c) Determine the correlation coefficient of X_1 and X_2. (5 pts)
   (d) Given a value τ, $0 \le \tau \le 2$, let's define a payoff function, $g(x_1, x_2)$, as following,
   $$g(x_1, x_2) = \tau \quad \text{if } x_2 \ge \tau, \text{ or}$$
   $$= x_2 \quad \text{if } x_2 < \tau.$$
   Find the expected payoff (representing as a function of τ). (5 pts)
   (e) Find the variance of payoff in (d). (5 pts)

5. Shewhart control chart is one of the statistical process control schemes, which uses "$\bar{x}$" as an indicator of whether the process is in control or out of control. For example, temperature is used to measure the output of a production process. When the process is in control, the mean of the process is $\mu = 50$ and the standard deviation is $\sigma = 0.5$.

   (a) If samples of size 6 are to be used, to monitor the process, please provide the LCL (lower control limit) and UCL (upper control limit) at 5% significance. (6 pts)
   (b) Is the process in control for a sample providing the following data? (2 pts)
       50.9　49.3　49.1　50.5　49.9　51.8
   (c) At the next sample point, the observed data are as following. Is the process still in control? (2 pts)
       50.9　50.5　50.7　49.3　50.1　48.2
   (d) At the next sample point following (c), the observed data are as following. Is the process still in control? (2 pts)
       50.7　50.6　50.5　49.2　50.0　48.1
   (e) Use the observed data in (b) and (c). Please conduct the test and determine whether the mean has changed or not? (4 pts)
   (f) Similarly, please conduct the test and determine whether the mean has changed or not between (c) and (d)? (4 pts)
   (g) Please provide your explanation about the contradiction (if you find any) for the in-control/out-of-control results in (b), (c), (d) and the mean-changed/mean-not-changed results in (e), (f). (5 pts)
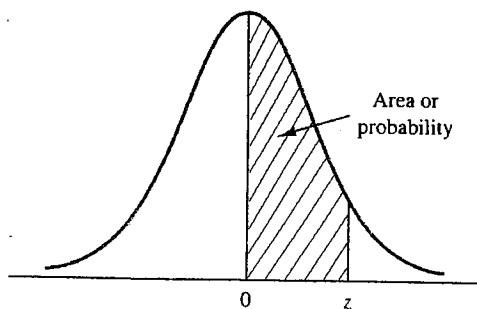
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Area or probability

Entries in the table give the area under the curve between the mean and z standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean and z is .3944.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990