參考用

Answering Problems

State with your reasoning or proofs. Please be precise and concise. No point will be graded if no explanation is provided. (答題請精準、簡捷，並皆須提示理由解釋或證明，否則不予計分。)

1. If for an unknown random variable X with $\mathbb{E}[X] = 35$, we also know that $P(X \geq 45) = 0.055$ and $P(X \leq 25) = 0.015$. Find the lower bound of $\text{var}(X)$. (8%)

2. Suppose $\{X_1, X_2, \cdots, X_n\}$ is a random sample drawn from a Poisson distribution with parameter λ,

   (a) Find the maximum likelihood estimator (MLE) for λ. (4%)

   (b) Please examine the consistency and sufficiency of the MLE. (8%)

3. Suppose the joint distribution of random variables X and Y is expressed as

$$f(x,y) = \begin{cases} \exp(-x), & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Please find $P(Y > 5 | X < 10)$. (結果以指數表達即可, 5%)

   (b) Find $\mathbb{E}[X + Y]$.(5%)

4. Given a set of random sampled data, {45, 11, 63, 59, 17, 4, 23, 28, 65, 42 }, from population, please find an approximately 95% confidence interval for the population median. (10%)

5. Let X and Y be two random variables, prove that

$$\text{var}(Y) = \mathbb{E}[\text{var}(Y|X)] + \text{var}(\mathbb{E}[Y|X]). \qquad (10\%)$$

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6. To test if t-test statistic is appropriate to use for a given population, we conduct an investigation with the following procedure. We randomly draw 1,000 *samples* from this population. Each of these samples has n observations drawn without replacement, and the t-statistic for each sample is calculated as

$$t_k = \frac{\bar{x}_k}{\sigma(x_k)/\sqrt{n}}, k = 1,\ldots,1000,$$

where $\{x_k\}$ are the observations in sample k, and 1,000 t-statistics $(t_1,\ldots,t_{1000})$ are derived. We then compare each of t_k to the critical values of t-test statistic associated with the two-tailed 5% significance level. Suppose 1,000 samples of size n are sufficient to make a correct inference, and we find that 92 of these 1,000 t-statistics are outside the 95% confidence interval. Based on such a finding, if we use t-statistic to test hypothesis on the observations drawn from this population, is it type I or type II error that we are likely to commit, and why? (6%)

7. Suppose we stand at time t=0 and consider the following model for the time-series dynamics of Y_t, t=1,2:

$$Y_1=\beta Y_0+u_1,$$
$$Y_2=\beta Y_1+u_2,$$

where the subscript represents time. Residual terms u_t satisfy

$$E(u_t)=0 \text{ for } t=1,2,$$
$$E(u_t^2)=\sigma^2 \text{ for } t=1,2,$$
$$E(u_1u_2)=\sigma_{12}\neq 0.$$

Y_0 is a known number at time t=0. Find E(Y_2) and Var(Y_2). (6%)

8. Suppose we need to estimate a linear regression model $Y_i=\beta X_i+\varepsilon_i$, where residual terms ε_i is independent and identically distributed, $E(\varepsilon_i)=0$, $Var(\varepsilon_i)=\sigma^2$, $Cov(\varepsilon_i,\varepsilon_j)=0$ if $i\neq j$, $i=1,\ldots,N$.

(a) Find the least square estimate of β (denoted as $\hat{\beta}$).(Note: Be sure to explain why your estimator gives the minimal sum of squared errors) (6%)

(b) Find $E(\hat{\beta})$ and $Var(\hat{\beta})$.Is $\hat{\beta}$ unbiased? (5%)

(c) Suppose now we have another estimator $\tilde{\beta} = \frac{\sum_{i=1}^{N} Y_i}{\sum_{i=1}^{N} X_i}$. Is $\tilde{\beta}$ unbiased? Which of $\tilde{\beta}$ and $\hat{\beta}$ is considered a better estimator and why? (6%)

(d) Suppose now $Var(\varepsilon_i)=\sigma_i^2$, where $\sigma_i^2\neq\sigma_j^2$ if $i\neq j$. Is $\hat{\beta}$ unbiased? What is Var($\hat{\beta}$)? (5%)

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9. We need to decide whether the returns of two different portfolios follow the same underlying Normal distribution. Suppose $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_m\}$ are the historical returns of portfolios X and Y, respectively. Please conduct the following tests with α% significance level. (Be sure to clearly specify the tested hypothesis, the used statistics, and the confidence interval.)
   (a) Test if the return standard deviations of portfolio X and Y are the same. (5%)
   (b) Suppose we conclude the return standard deviations for two portfolios are the same from (a). Test if the means of returns of portfolio X and Y are the same. (5%)

10. Consider the following simultaneous equation system describing the mutual influence between the equilibrium price (p) and the quantity (q) of a product over time:

$$p_t=\beta_1 q_t+\beta_2 x_t+u_t, \quad (1)$$

$$q_t=\gamma p_t+v_t, \quad (2)$$

where subscript t represents time, x_t is an exogenous variable that affects price, and u_t and v_t are residual terms satisfying

$$E(u_t)=E(v_t)=0$$

$$E(u_t^2)=\sigma_u^2,\ E(v_t^2)=\sigma_v^2$$

$$E(u_t v_t)=E(u_t x_t)=E(v_t x_t)=0.$$

Show $Cov(q,u)\neq 0$ and $Cov(p,v)\neq 0$. (6%)