所別：通訊工程學系碩士班 乙組(通訊網路)(一般生)　科目：工程數學　共 3 頁 第 1 頁
本科考試禁用計算器　＊請在試卷答案卷（卡）內作答

以下共分 A、B和 C 三部份，每一部份 50 分，任選兩部份做答。請在答案卷最前面先註明您選答那兩部份，未註明者，不得對改卷所挑選之部份有異議。

Part A 機率 (50分)

1. **(10%)** The binomial random variable X has probability mass function (PMF) $P_X(x)=\binom{5}{x}(1/2)^5$. Let μ_X and σ_X denote the expected value and standard deviation of X, respectively.
   (1) (5%) Please find μ_X.
   (2) (5%) Please find $P[\mu_X-\sigma_X \le X \le \mu_X+\sigma_X]$.
2. **(10%)** X is a uniform random variable with parameters -5 and 5. Given the event $\{A=|X|\le 2\}$.
   (1) (5%) Please find the conditional PDF $f_{X|A}(x)$.
   (2) (5%) Please find the conditional variance Var[$X|A$].
3. **(5%)** Random variables X and Y have the joint PDF
   $$f_{X,Y}(x,y)=\begin{cases}2 & 0\le y\le x\le 1,\\ 0 & \text{otherwise.}\end{cases}$$
   (1) (5%) Please find $f_W(w)$, where $W=X/Y$.
4. **(15%)** Let the random vector $\mathbf{X}=[x_1 \quad x_2]^T$. The PDF of $\mathbf{X}$ is
   $$f_{\mathbf{X}}(\mathbf{x})=\begin{cases}2 & \mathbf{x}\ge \mathbf{0},\ x_1+x_2\le 1,\\ 0 & \text{otherwise.}\end{cases}$$
   (1) (5%) Please find the expected value vector $E[\mathbf{X}]$.
   (2) (5%) Please find the correlation matrix $\mathbf{R_X}$.
   (3) (5%) Please find the covariance matrix $\mathbf{C_X}$.
5. **(10%)** Moment generating function.
   (1) (5%) X is an exponential random variable with moment generating function $\phi_X(s)=\frac{\lambda}{\lambda-s}$. Please give the general expression for the nth moment of X.
   (2) (5%) Let $K_1,\ldots,K_n$ denote a sequence of iid Bernoulli (p) random variables, where each K_i has probability mass function
   $$P_K(k)=\begin{cases}1-p & k=0,\\ p & k=1,\\ 0 & \text{otherwise.}\end{cases}$$
   Let $M=K_1+K_2+\cdots+K_n$. Find the moment generating function $\phi_M(s)$.

參考用

注意：背面有試題

Part B 離散數學 (50 分)

1. (6%) Determine if each of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
   (a) (2%) $f(x) = -3x + 4$. (True/False)
   (b) (2%) $f(x) = -3x^2 + 7$. (True/False)
   (c) (2%) $f(x) = (x+1)/(x+2)$. (True/False)

2. (6%) Find all solutions, if any, to the system of congruences.
   $x \equiv 5 \pmod 6$
   $x \equiv 3 \pmod{10}$
   $x \equiv 8 \pmod{15}$

3. (6%) Let a_n be the n-th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$, constructed by including the integer k exactly k times. Show that $a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.

4. (6%) Please answer the following questions in regard to *well-ordering principle.*
   (a) (3%) Define or formally explain the term of "well-ordering principle".
   (b) (3%) Please use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$.

5. (6%) Consider a sequence defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \times a_{n-2}$, for $n = 2, 3, 4, \ldots$
   (a) (3%) Design a recursive algorithm to find the n-th term of this sequence.
   (b) (3%) Design an iterative algorithm to find the n-th term of this sequence.

6. (10%) Please answer the following questions in regard to *graphs*:
   (a) (2%) Define or formally explain the term of "bipartite graph".
   (b) (2%) Define or formally explain the term of "isomorphic simple graph".
   (c) (2%) How many non-isomorphic connected bipartite simple graphs are there with four vertices ?
   (d) (2%) How many non-isomorphic simple connected graphs with five vertices are there, when no vertex of degree is more than two?
   (e) (2%) How many non-isomorphic simple connected graphs with five vertices are there, when being non-planar?

7. (10%) Solve the simultaneous recurrence relations
   $a_n = 3a_{n-1} + 2b_{n-1}$
   $b_n = a_{n-1} + 2b_{n-1}$
   with $a_0 = 1$ and $b_0 = 2$.

參考用

注意：背面有試題

Part C 線性代數 (50 分)

1. (7%) Let A and B be similar matrices. Prove that $\det(A-\lambda I)=\det(B-\lambda I)$, where λ is any scalar and det(.) is the determinant of the indicated matrix.

2. (7%) Let A be a nonsingular $n\times n$ matrix with $n>1$, and let $\det(A)$ and $\text{adj}(A)$ be the determinant and the adjoint of the matrix A respectively. Prove that $\det(\text{adj}(A)) = (\det(A))^{n-1}$.

3. (8%) Let $E=\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ and $F=\{\mathbf{b}_1,\mathbf{b}_2\}$, where

$$\mathbf{u}_1=\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad \mathbf{u}_2=\begin{bmatrix}1\\2\\1\end{bmatrix},\quad \mathbf{u}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix},$$

and

$$\mathbf{b}_1=(1,-1)^T,\quad \mathbf{b}_2=(2,-1)^T.$$

For the linear transformation $L(x)=(x_1+x_2,x_1-x_3)^T$ from $\mathbb{R}^3$ to $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F.

4. (8%) Let $A=(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_5)$ be a 4×5 matrix, and let U be the reduced row echelon form of A.

$$U=\begin{bmatrix}1&0&2&0&-1\\0&1&3&0&-2\\0&0&0&1&5\\0&0&0&0&0\end{bmatrix},\ \mathbf{a}_1=\begin{bmatrix}2\\1\\-3\\-2\end{bmatrix},\ \mathbf{a}_2=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}.$$

(a) (3%) Find a basis for $N(A)$, the null space of A.

(b) (5%) Given that x_0 is a solution of $Ax=b$, where $b=(0,5,3,4)^T$ and $x_0=(3,2,0,2,0)^T$. Please determine the other remaining column vectors of A.

5. (10%) Consider the inner product space $C[0,1]$ with inner product defined by $<f,g>=\int_0^1 f(x)g(x)dx$. Let S be the subspace spanned by two vectors 1 and $2x-1$.

(a) (2%) Show that 1 and $2x-1$ are orthogonal.

(b) (8%) Find the best squares approximation to $\sqrt{x}$ by a linear function function from the subspace S.

6. (10%) Use the eigenvalues approach to find the solution to the initial value problem of the following linear differential equations.

$$\begin{cases} y_1''=-2y_2+y_1'+2y_2' \\ y_2''=2y_1+2y_1'-y_2' \\ y_1(0)=1,\ y_2(0)=0,\ y_1'(0)=-3,\ y_2'(0)=2\end{cases}$$

參考用