1. Please solve $4y'' + 36y = csc3x$. (5%)

2. Give the conditions on $f(t)$ and $g(t)$ under which their Laplace transform exists and then prove $\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$, CLEARLY AND RIGOROUSLY, where $\mathcal{L}\{f\} = F(s)$ is the Laplace transform of $f(t)$ and $*$ denotes the convolution operator. (10%)

3. Suppose that a hawk P at the point $(a, 0)$ spots a pigeon Q at the origin flying along the positive y-axis at speed v. The hawk immediately flies toward the pigeon at a speed w.
(a) What will be the flight path of the hawk if $w > v$? (10%)
(b) If $w > v$, can the hawk catch the pigeon? where can it do that? (5%)
(c) If $w = v$, what will be the flight path of the hawk? (5%)
(Hint: $\int \frac{du}{\sqrt{u^2+v^2}} = \ln(u + \sqrt{u^2+v^2}) + c$).

4. (a) Find two unit vectors orthogonal to $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. (5%)
(b) If the following matrix is invertible, find the inverse. (5%)

$$\begin{pmatrix} 1 & -3 & 0 & -2 \\ 3 & -12 & -2 & -6 \\ -2 & 10 & 2 & 5 \\ -1 & 6 & 1 & 3 \end{pmatrix}$$

5. (a) Find the rank, the nullity and the kernel of the matrix

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix}. \qquad (5\%)$$

(b) Describe geometrically the linear transformation $T: R^3 \to R^3$ given by $T\mathbf{x} = A\mathbf{x}$.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (5\%)$$

6. Diagonalize $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. (10%)

參考用

15% 7.

5% a/ Show that $\cosh z = \cosh x \cos y + i \sinh x \sin y$, where $z=x+iy$.

5% b/ Show that $\sin^{-1} z = -i \ln [iz + (1-z^2)^{1/2}]$.

5% c/ Find the principle value of $(1+i)^{1-i}$.

10% 8. The Fourier transform is defined as

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt, \qquad -\infty < w < \infty$$

Find the Fourier transform of the following function:

$$f(t) = \frac{1}{4+t^2}$$

10% 9. The Laplace transform is defined as

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt$$

Use residue theorem to find the inverse Laplace transform f(t) of the following:

$$F(s) = \frac{s}{s^2+1}$$