This exam contains 6 questions, adding up to 100 points. Please read the questions carefully and write down the detail calculations with solid logic flows and sensible arguments. Blind guesses or orphan answers are not credited.

1. Quadratic Form (15%). In the three-dimensional space, a quadratic form takes the following expression,

$$Q = 4x^2 + 4y^2 + 5z^2 + 2yz + 2xz\,,$$

with the constraint $x^2 + y^2 + z^2 = 1$. Find the vector $\vec{r} = (x, y, z)$ where the quadratic form reaches its minimum.

2. Matrix Function (15%). A quantum system composed of three states with mutual tunneling between them is captured by the following Hamiltonian,

$$H = \begin{pmatrix} 0 & \gamma & \gamma \\ \gamma & 0 & \gamma \\ \gamma & \gamma & 0 \end{pmatrix},$$

where the tunneling amplitude γ is chosen to be real. Compute the corresponding partition function $Z = \mathrm{tr}[\exp\,(-H/kT)]$ of the quantum system.

3. Relaxation Dynamics (15%). A gating variable is originally are rest, $x(t < 0) = 0$, and stimulated by an external Heaviside function $\Theta(t)$, defined as $\Theta(t < 0) = 0$ and $\Theta(t \geq 0) = 1$. Its temporal evolution can be described by the relaxation dynamics,

$$\frac{\tau^2}{t}\frac{dx}{dt} + x = \Theta(t).$$

Here $\tau > 0$ is the relaxation time. Find the time evolution of the gating variable $x(t)$.

4. Quantum Oscillator (15%). The eigenstate wave functions of the simple harmonic oscillator take the following form,

$$\psi_n(x) = \exp\left(-\frac{x^2}{2}\right) H_n(x),$$

where H_n is the Hermite polynomial defined through the generating function $g(x, t)$,

$$g(x, t) = \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

Find the eigenstate wave function $\psi_3(x)$ of the quantum oscillator.

5. Vibrating String (20%). An elastic string with both ends fixed at $x = \pm L$ satisfies the one-dimensional wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Here $y(x,t)$ denotes the displacement of the string from its equilibrium position and c is the wave speed on the elastic string. Initially, the center of the string (at $x = 0$) is pulled to the height h (forming a triangular initial configuration) and then released from rest at $t = 0$. Find the solution of the vibrating string $y(x,t)$ afterward.

6. Klein-Gordon Equation (20%). A spinless relativistic particle described by the scalar field $\phi(\vec{r})$ satisfies the Klein-Gordon equation,

$$\nabla^2 \phi = m^2 \phi.$$

Show that the scalar field $\phi(\vec{r})$ is unique in any given volume V bounded by the surface $S = \partial V$ with specified Neumann boundary conditions. [*hint:* Green's theorem can provide helpful identity to facilitate the proof.]

注意:背面有試題