## 台灣聯合大學系統 108 學年度碩士班招生考試試題

類組:物理類 科目:應用數學(2001)

共 乙 頁第 / 頁

※請在答案卷內作答

Unless explicitly stated, the meaning of the symbol i is

$$i = \sqrt{-1}. (1)$$

1. Evaluate the line integral in the three-dimensional space, with (x, y, z) being the Cartesian coordinate of a point in this space,

$$I = \oint_C \left[ \left( e^{-x^2} - yz \right) dx + \left( e^{-y^2} + xz + 2x \right) dy + e^{-z^2} dz \right], \tag{2}$$

where the closed contour C is the circle

$$x = \cos\theta, \ y = \sin\theta, \ z = 1, \tag{3}$$

oriented in the direction of increasing  $\theta$  ( $0 \le \theta < 2\pi$ ). [15 points]

- 2. Evaluate the following one-dimensional integrals:
  - (a) With  $\omega$  and  $\tau$  being real, and  $\epsilon$  being real and positive, perform the integral,

$$S(\tau) = \lim_{\epsilon \to 0^{+}} \left( i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega + i\epsilon} \right). [10 \text{ points}]$$
 (4)

(b) With x and p being real variables, and m being a real parameter, perform the integral

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\mathrm{e}^{-ipx}}{p^2 + m^2}.$$
 [10 points] (5)

3. In this problem, we will find solutions to the non-linear differential equation,

$$\frac{\partial^2 \varphi}{\partial \sigma \partial \rho} - \sin(\varphi) = 0, \tag{6}$$

where  $\varphi$  is a real function that depends on two real variables,

$$\varphi = \varphi(\sigma, \rho). \tag{7}$$

(a) Let  $\varphi_0$  be a given solution to Eq. (6). Show that another solution,  $\varphi_1$ , to Eq. (6) can be obtained by solving the equations

$$\frac{1}{2}\frac{\partial}{\partial\sigma}\left(\varphi_1-\varphi_0\right) = a\sin\left[\frac{1}{2}\left(\varphi_1+\varphi_0\right)\right],\tag{8}$$

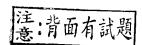
$$\frac{1}{2}\frac{\partial}{\partial\rho}\left(\varphi_1+\varphi_0\right) = \frac{1}{a}\sin\left[\frac{1}{2}\left(\varphi_1-\varphi_0\right)\right],\tag{9}$$

where a is a real parameter. [5 points]

(b) It is obvious that

$$\varphi_0 = 0, \tag{10}$$

is a solution to Eq. (6). Take this "trivial" solution, and use Eqs. (8) and (9) to find a non-trivial solution,  $\varphi_1$ . [10 points]



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共\_三頁第2頁

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4. A complex function is defined by the following expression:

$$f(z) = \int_0^\infty e^{-zt} dt,\tag{11}$$

where t is real

- (a) Find the domain where  $f\left(z\right)$  exists and analytic. [10 points]
- (b) Determine the analytic continuation of f(z) over the entire z-plane except for Re(z)=0. [10 points]
- 5. A  $3 \times 3$  matrix is given as

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \tag{12}$$

- (a) Compute the eigenvalues of A. [10 points]
- (b) Find the general solution to the following differential equation. [10 points]

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},\tag{13}$$

where  $\mathbf{x}$  is a  $3 \times 1$  vector.

6. An antiunitary operator U that maps x to y in complex Hilbert space satisfies the following relation

$$\langle Ux|Uy\rangle = \langle y|x\rangle.$$
 (14)

Prove the following relations.

- (a)  $U^2$  is unitary. [5 points]
- (b) The product of U and complex conjugate operator  $\mathcal{K},$  i.e.,  $U\mathcal{K},$  is unitary. [5 points]