Unless explicitly stated, the meaning of the symbol i is

$$i = \sqrt{-1}. \tag{1}$$

1. Evaluate the line integral in the three-dimensional space, with (x, y, z) being the Cartesian coordinate of a point in this space,

$$I = \oint_C \left[\left(e^{-x^2} - yz\right) dx + \left(e^{-y^2} + xz + 2x\right) dy + e^{-z^2} dz\right], \tag{2}$$

where the closed contour C is the circle

$$x = \cos\theta,\ y = \sin\theta,\ z = 1, \tag{3}$$

oriented in the direction of increasing θ $(0 \le \theta < 2\pi)$. [15 points]

2. Evaluate the following one-dimensional integrals:

(a) With ω and τ being real, and ϵ being real and positive, perform the integral,

$$S(\tau) = \lim_{\epsilon \to 0^+} \left(i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega + i\epsilon}\right). \text{ [10 points]} \tag{4}$$

(b) With x and p being real variables, and m being a real parameter, perform the integral

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{-ipx}}{p^2 + m^2}. \text{ [10 points]} \tag{5}$$

3. In this problem, we will find solutions to the non-linear differential equation,

$$\frac{\partial^2 \varphi}{\partial\sigma\partial\rho} - \sin(\varphi) = 0, \tag{6}$$

where φ is a real function that depends on two real variables,

$$\varphi = \varphi(\sigma, \rho). \tag{7}$$

(a) Let φ_0 be a given solution to Eq. (6). Show that another solution, φ_1, to Eq. (6) can be obtained by solving the equations

$$\frac{1}{2}\frac{\partial}{\partial\sigma}(\varphi_1 - \varphi_0) = a \sin\left[\frac{1}{2}(\varphi_1 + \varphi_0)\right], \tag{8}$$

$$\frac{1}{2}\frac{\partial}{\partial\rho}(\varphi_1 + \varphi_0) = \frac{1}{a} \sin\left[\frac{1}{2}(\varphi_1 - \varphi_0)\right], \tag{9}$$

where a is a real parameter. [5 points]

(b) It is obvious that

$$\varphi_0 = 0, \tag{10}$$

is a solution to Eq. (6). Take this "trivial" solution, and use Eqs. (8) and (9) to find a non-trivial solution, φ_1. [10 points]

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4. A complex function is defined by the following expression:

$$f(z) = \int_0^{\infty} e^{-zt} dt, \tag{11}$$

where t is real

(a) Find the domain where $f(z)$ exists and analytic. [10 points]

(b) Determine the analytic continuation of $f(z)$ over the entire z-plane except for $\mathrm{Re}(z) = 0$. [10 points]

5. A 3×3 matrix is given as

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \tag{12}$$

(a) Compute the eigenvalues of A. [10 points]

(b) Find the general solution to the following differential equation. [10 points]

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{13}$$

where $\mathbf{x}$ is a 3×1 vector.

6. An antiunitary operator U that maps x to y in complex Hilbert space satisfies the following relation

$$\langle Ux|Uy\rangle = \langle y|x\rangle. \tag{14}$$

Prove the following relations.

(a) U^2 is unitary. [5 points]

(b) The product of U and complex conjugate operator $\mathcal{K}$, i.e., $U\mathcal{K}$, is unitary. [5 points]