科目：工程數學C(3005)

校系所組：中央大學電機工程學系(電子組)
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所(甲組、乙組)
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- 請將答案依下圖所示由上而下依序寫在答案卷的作答區的第一頁。
- 只要填寫考題所要求的答案，請勿附加計算過程。

從此處開始寫起
一、
二、
三、
四、
五、(一)⋯　(二)⋯
⋮

一、 (5%) If $F(s)$ is the Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, find the inverse Laplace transform $\mathcal{L}^{-1}\{F(as+b)\}$ in terms of $f(t)$, where $a > 0$ and $b \neq 0$.

二、 (5%) Solve

$$y'(t) = y(t) + 4\int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau, \quad y(0) = 1.$$

三、 (5%) Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}.$$

Compute e^{At}.

四、 (5%) Consider the non-homogeneous linear system $\underline{x}' = A\underline{x} + e^{\alpha t}\underline{v}$, where $\underline{x}$ is a vector consisting of functions in t, α is not an eigenvalue of A, and $\underline{v} \neq \underline{0}$ is a constant vector. Find a particular solution of the system, in terms of A, α, $\underline{v}$ and t.

五、 (10%)

（一） (5%) Determine the Fourier series coefficients (a_n, b_n) of the function $f(t) = t \cdot u(t)$ expanded over the interval $(-\pi, 2\pi)$, where $u(t)$ is the unit-step function.

（二） (5%) If the coefficients (a_n, b_n) from 五、（一） are also the Fourier series coefficients of some function expanded over the interval $(-2\pi, 4\pi)$, find the function in terms of $f(t)$.

六、 (10%) Solve the following boundary value problem for $f(x,t)$ with $x > 0$ and $0 < t < 10$

$$\frac{\partial^2}{\partial t^2} f(x,t) = 3\frac{\partial}{\partial x} f(x,t),$$

$$\left.\frac{\partial}{\partial t} f(x,t)\right|_{t=0} = \left.\frac{\partial}{\partial t} f(x,t)\right|_{t=5} = \left.\frac{\partial}{\partial t} f(x,t)\right|_{t=10} = 0, \quad f(0,t) = 4\cos(\pi t)$$

注意：背面有試題

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參考用

七、(12%) Given $y_1(x) = x^r$ is one solution of the homegeneous 2nd order linear differential equation $x^2y'' - 5xy + 9y = 0$.

（一） (2%) Derive its characteristic equation in terms of parameter r.

（二） (3%) Let $y_2(x) = v(x)y_1(x)$ be another linearly independent solution. Determine the governing differential equation of $v(x)$.

（三） (3%) Find $v(x)$ by solving the differential equation in 七、（二）.

（四） (4%) Apply the method of variation of parameters to find a particular solution of $y'' - \frac{5}{x}y' + \frac{9}{x^2}y = x^2$

八、(8%) Solve the differential equation $(x^2-1)y'' - 6xy' + 12y = 0$ by power series of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$.

（一） (2%) Find the recurrence relation of c_n

（二） (4%) Find the two linearly independent solutions. Write the first three nonzero terms of each series if it is an infinite series.

（三） (2%) Find the guaranteed radius of convergence.

九、(10%) A system of linear equations has unknown coefficients which can be expressed with the real variable a. The system is as follows

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1-a^2 & 1 \\ 0 & 0 & 1+a^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Please determine the value of a for the system to have nontrivial solutions.

十、(7%) Let A be an $n \times n$ matrix. Assume $\sigma_1 \geq \cdots \geq \sigma_n$ are singular values of A. For any $\underline{x} \neq \underline{0}$ (element-wise not equal to), what is the relationship between $\sigma_1 \|\underline{x}\|_2$, $\sigma_n \|\underline{x}\|_2$, and $\|A\underline{x}\|_2$, where $\|\underline{x}\|_2$ denotes the Euclidean norm of vector $\underline{x}$? Justify your answer algebraically.

十一、(8%) Given $n \times n$ positive definite matrices A and B, for any $\underline{x} \neq \underline{0}$, derive the expression to find the smallest value of $\det\left(\frac{\underline{x}^\top A \underline{x}}{\underline{x}^\top B \underline{x}}\right)$. What is this smallest value? Find the expression for $\underline{x}$, or function thereof, to achieve this minimum value.

注：背面有試題

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十二、(15%) Let $\mathcal{P}_3$ be the set of all polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$, where a_0, a_1, a_2, and a_3 are real numbers. Assume $T : \mathcal{P}_3 \to \mathcal{P}_3$ is a linear transformation with

$$T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = (a_0 + a_3) + (a_1 + a_0)x + (a_2 + a_1)x^2 + (a_3 + a_2)x^3$$

（一） (6%) Find the range and null space of T.

（二） (4%) Assume any polynomial $p(x) \in \mathcal{P}_3$ can be represented as

$$p(x) = x_1 \cdot 1 + x_2 \cdot (1 + x) + x_3 \cdot (1 + x + x^2) + x_4 \cdot (1 + x + x^2 + x^3)$$

and its corresponding polynomial $T(p(x))$ is represented as

$$T(p(x)) = y_1 \cdot 1 + y_2 \cdot (1 + x) + y_3 \cdot (1 + x + x^2) + y_4 \cdot (1 + x + x^2 + x^3).$$

Please find the corresponding matrix M such that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

（三） (5%) Please find all polynomials that map to $1 + 2x + 2x^2 + x^3$. That is, please find all $p(x)$ such that $T(p(x)) = 1 + 2x + 2x^2 + x^3$.