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※請在答案卷內作答　計算題，請寫出計算過程。

1. [5%] Find the general solution for $y'' + 5y' + 4y = 10e^{-3x}$

2. [5%] Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem
$y'' + \lambda y = 0, \quad y(0) = 0, \ y(l) = 0$

3. [5%] Use Laplace Transform method to solve the ODE
$y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0$

4. [5%] Find the inverse Laplace transform for $\mathcal{L}^{-1}\left(\frac{1}{(s-1)(s+2)}\right)$

5. [5%] Find the Fourier transform for $\mathrm{f(t) = e^{-a|t|}, a > 0}$

6. Let $\mathrm{P}_3(R)$ denote the set of all functions $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where coefficients $a_0, a_1, a_2, a_3 \in R$. Define linear transformation $\mathrm{T}: \mathrm{P}_3(R) \to \mathrm{P}_3(R)$ as follows: $\mathrm{T}(f(x)) = df/dx$.
   (a) [3%] Let $\beta = \{1, x, x^2, x^3\}$ be an ordered basis of $\mathrm{P}_3(R)$. Write down the matrix representation $A = [\mathrm{T}]_\beta$.
   (b) [2%] Is A invertible?
   (c) [5%] Is A diagonalizable?
   (d) [10%] Now consider $f(x), g(x) \in \mathrm{P}_3(R)$ as functions defined on the close interval $[-2,2]$. Define inner product in the standard way: $\langle f, g \rangle = \int_{-2}^{2} f(x)g(x)dx$. Find a function $u(x) \in \mathrm{P}_3(R)$ such that $\langle u, f \rangle = 0$ for any given second-order polynomial $f(x) = a_0 + a_1x + a_2x^2$.

7. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
   (a) [5%] Find the characteristic polynomial of A and express its two roots in terms of θ.
   (b) [5%] What are all possible $\theta \in [-\pi, \pi]$ such that $A^{2019} = I$?

注意：背面有試題

※請在答案卷內作答

8. [10%] Let $u_1 = (1,1,0)^T, u_2 = (0,1,1)^T$, and $u_3 = (0,0,1)^T$. Find $\alpha, \beta \in R$ such that $J = \|u_3 - \alpha u_1 - \beta u_2\|^2$ is minimized. Here, $\|\cdot\|$ denotes the standard l^2-norm.

9. [10%] Let $u_1, u_2, \ldots, u_M$ be column vectors in R^N. Let A be a matrix of size $M \times M$ and its elements are defined as $A_{ij} = u_i^T u_j$, where $1 \le i, j \le M$. Prove that for any $x = (x_1, x_2, \ldots, x_M)^T \in R^M$, we have $x^T A x \ge 0$. Under what conditions does $x^T A x = 0$ imply $x = 0$?

10. [10%] Evaluate the integral $\int_0^{1+i} (x - y + ix^2) dz$

(a) along the straight line from $z = 0$ to $z = 1 + i$.

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

11. [10%] Evaluate $I = \oint_C \frac{dz}{z^2(z-2)(z-4)}$, where C is the rectangle joining the points (-1,-1), (3,-1), (3,1) and (-1,1) in clockwise sense in the complex plane.

12. [5%] Evaluate by residue method the integral $\oint_C z^{-4} \sin(z)\, dz$; C: $|z| = 1$ taken in counter-clockwise sense in the complex plane.