類組:電機類 科目:工程數學 B(3004)

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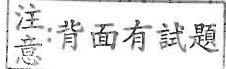
※請在答案卷內作答

Note: Detailed derivations are required to obtain a full score for Problem 2 to Problem 10.

- 1. (15%) Among the 10 statements below, only 5 are true and the other 5 are false. Find out which 5 are true. (You are not obligated to give explanations, but will get zero point if listing more than 5 of them).
  - (a) Let V be a vector space and  $S \subseteq V$  be a subset. Then,  $\operatorname{span}(S)$  is the intersection of all subspaces of V that contain S.
  - (b) Let  $T: V \to W$  be a linear transformation. Let  $S = \{v_1, v_2, ..., v_n\}$  be a subset of V. If S is linearly dependent, its image T(S) is also linearly dependent.
  - (c) The basis of any vector space uniquely exists.
  - (d) Let  $T: V \to W$  be a linear transformation. If T is invertible, then  $\dim(V) = \dim(W)$ .
  - (e) Let  $A \in M_{m \times n}(\mathbb{R})$  be an arbitrary matrix. If m < n, then  $rank(A) > rank(A^t)$ .
  - (f) Assume that  $A \in M_{m \times n}(\mathbb{R})$  and  $b \in M_{m \times 1}(\mathbb{R})$ . Let  $x_1$  and  $x_2$  be two column vectors in  $\mathbb{R}^n$ . If  $x_1 \neq x_2$  and  $Ax_1 = b = Ax_2$ , then the system of linear equations Ax = b has infinitely many solutions.
  - (g) Let A and B be square matrices of the same size. If AB = O, then  $R(L_B) \supseteq N(L_A)$ . (Remarks:  $L_A$  and  $L_B$  denote the linear transformation of matrix multiplication from the left.)
  - (h) Let A and B be square matrices of the same size. If AB = A, then B = I.
  - (i) Let A be a square matrix and  $r \in \mathbb{R}$ . Then,  $\det(rA) = r \det(A)$ .
  - (j) Assume that  $A \in M_{3\times 3}(\mathbb{C})$  and  $A^tA = -I$ . Then, the entries in A cannot all be real numbers.
- 2. (10%) Define a linear transformation  $T : \mathbb{R}^3 \to \mathbb{R}^3$  by T((1,0,0)) = (0,1,0), T((0,1,0)) = (0,0,1), and T((0,0,1)) = (1,0,0).
  - (a) (5%) Find a vector  $u = (u_x, u_y, u_z)$  such that T(u) = u and  $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$ .
  - (b) (5%) Is  $T : \mathbb{R}^3 \to \mathbb{R}^3$  one-to-one and onto? Why or why not?
- 3. (15%) Let  $W_1, \ldots, W_k$  be subspaces of a vector space V. The **direct sum** V of  $W_1, \ldots, W_k$  is defined if the following two conditions hold.

$$V = \sum_{i=1}^{n} W_i$$
 and  $W_j \cap \sum_{i \neq j} W_i = \{0\} \ \forall j \ (1 \le j \le k)$ 

If the two conditions hold, then V is denoted by  $V = W_1 \oplus \ldots \oplus W_k$ . Prove or disprove (by providing a counterexample) of the following statements.





## 台灣聯合大學系統 106 學年度碩士班招生考試試題

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※請在答案卷內作答

- (a) (7%) If  $V = W_1 \oplus \ldots \oplus W_k$ . Then, for any distinct i and j,  $W_i$  and  $W_j$  intersect at exactly the zero vector.
- (b) (8%) If  $V = \sum_{i=1}^{n} W_i$ , and  $W_i$  and  $W_j$  intersect at exactly the zero vector for any distinct  $i, j \ (1 \le i, j \le k)$ . Then V is the direct sum of  $W_1, \ldots, W_k$ .
- 4. (10%) Let V be a finite-dimensional complex inner product space and T: V → V be a linear operator. T is normal if and only if TT\* = T\*T, where T\* is the adjoint of T. Moreover, T is nilpotent if there exists n ∈ N such that T<sup>n</sup> is the zero operator. Prove the following statement. If T is both normal and nilpotent, then T is the zero operator itself.
- 5. (7%) Prove that if  $\Theta$  is a random variable from the interval  $[0, 2\pi]$ , then the dependent variables  $X = \sin \Theta$  and  $Y = \cos \Theta$  are uncorrelated.
- **6.** (8%) Let X be a random variable; show that for  $\alpha > 1$  and t > 0,  $P(X \ge \frac{\ln \alpha}{t}) \le \frac{M_X(t)}{\alpha}$ , where  $M_X(t)$  is the moment generating function of X.
- 7. (10%) First a point Y is selected at random from the interval (0, 1). Then another point X is selected at random from the interval (Y, 1). Find the probability density function of X.
- 8. (7%) A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head.
  - (a) (4%) Is she correct?
  - (b) (3%) Does it matter if it is a fair coin or an unfair coin? Compute the exact probabilities for each of the scenarios described above given that we know the coin is a fair coin.
- 9. (8%) Consider four independent rolls of a 6-sided die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y?
- 10. (10%) Alice passes through four traffic lights on her way to work, and each light is equally-likely to be green or red, independent of the others.
  - (a) (5%) What is the mean and the variance of the number of red lights that Alice encounters?
  - (b) (5%) Suppose that each red light delays Alice by exactly two minutes. What is the variance of the time by which Alice is delayed by the red lights?

注:背面有試題

