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- 本測驗試題為複選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題完全答對得五分，答錯不倒扣。

Notation: In the following questions, underlined letters such as $\underline{a}, \underline{b}$, etc. denote column vectors of proper length; boldface letters such as $\mathbf{A}, \mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$, and $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\mathbb{R}$ is the usual set of all real numbers. $\langle \underline{a}, \underline{b} \rangle$ denotes the inner product of vectors $\underline{a}$ and $\underline{b}$. If X is a discrete random variable, then the probability mass function (PMF) of X is denoted by $p_X(x)$; if X is a continuous random variable, it is always assumed that X has a probability density function (PDF), denoted by $f_X(x)$. $\mathbb{E}[X]$ means the expected value of a random variable X. $\Pr()$ denotes the probability measure in a probability space.

一、 Let V and W be finite dimensional real vector spaces with ordered bases β and γ. Assume that T and U are linear transformations from vector space V into W. Which of the following statements are true?

(A) For any scalar $a \in \mathbb{R}$, $aT + U$ is a linear transformation.

(B) If n is the dimension of vector space V and m is the dimension of vector space W, then the matrix $\mathbf{A}$ to represent T relative to the bases β and γ is an $(n \times m)$ matrix.

(C) If $W = V$, T has an inverse linear transformation T^{-1}.

(D) If T is onto, then the nullity of T equals 0.

(E) None of the above are true.

二、 A generalized quadratic equation in two variables x and y is an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, where a, b, c, d, e, f are some real constants. Let $\underline{z} = [x\ y]^\top$; then we can represent the generalized quadratic equation as $\underline{z}^\top \mathbf{A} \underline{z} + \underline{g}^\top \underline{z} + f = 0$ for some matrix $\mathbf{A}$ and vector $\underline{g}$. Assuming $\mathbf{A}$ is symmetric, which of following statements are true?

(A) $\mathbf{A}$ is always orthogonally diagonalizable.

(B) Suppose that $\mathbf{A}$ has an eigenvalue λ with multiplicity k, then the eigenspace associated with λ can have dimension less than k.

(C) $\mathbf{A}$ can be factored into a matrix-product $\mathbf{QR}$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

(D) If $\det(\mathbf{A}) = 0$, the solutions (x, y) to the corresponding generalized quadratic equation form a parabola on the two dimensional Cartesian plane.

(E) None of the above are true.

三、 V and W are both subspaces of a vector space U. Let $\mathcal{V} = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ and $\mathcal{W} = \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_m\}$ be sets of linearly independent vectors, which span V and W, respectively. Which of the following statements are true?

(A) The set-union of V and W is a subspace of U.

(B) The dimension of the set-union of V and W is equal to $k + m$.

(C) The intersection of $\mathcal{V}$ and $\mathcal{W}$ is a linearly independent set.

(D) Every vector in the set-union of V and W is a certain linear combination of elements in $\mathcal{V}$ and $\mathcal{W}$.

(E) None of the above are true.

四、 Let $\mathbf{A}$ be an $(m \times n)$ matrix, which can be factored into a matrix-product $\mathbf{QR}$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix. Which of following statements are true?

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(A) $m \geq n$

(B) The system of linear equations $\mathbf{A}\underline{x} = \underline{b}$ must be a consistent system for any vector $\underline{b} \in \mathbb{R}^m$.

(C) The right null space of $\mathbf{A}$ contains only the all-zero vector.

(D) If $m = n$, then $\mathbf{A}$ is diagonalizable.

(E) None of the above are true.

五、 Let

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}.$$

Which of the following statements are true?

(A) $\mathbf{P}$ has three distinct eigenvalues.

(B) $\mathbf{P}$ is diagonalizable.

(C) $\mathbf{P}^4 = \begin{bmatrix} 31 & 30 & -60 \\ 15 & 16 & -30 \\ 15 & 15 & 29 \end{bmatrix}$

(D) $\mathbf{P}$ has an LU decomposition as $\mathbf{P} = \mathbf{LU}$, and the elements in the first row of $\mathbf{U}$ are integers.

(E) None of the above are true.

六、 Which of the following statements are true?

(A) If the vectors $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_4$, and $\underline{v}_5$ span $\mathbb{R}^4$, then $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ must form a basis for $\mathbb{R}^4$.

(B) If the rank of a (7×11) real matrix $\mathbf{A}$ is 3, then the right null space of $\mathbf{A}$ must be eight dimensional over $\mathbb{R}$.

(C) There exists a noninvertible (2×2) matrix $\mathbf{A}$ that is similar to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(D) If V is the set of all (3×3) real matrices $\mathbf{A}$ such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of $\mathbf{A}$, then V is a subspace of $\mathbb{R}^{2\times 2}$.

(E) None of the above are true.

七、 Let $\mathbf{A}$ and $\mathbf{B}$ be any $(n \times n)$ real matrices. Which of the following are true?

(A) Eigenvalues of $\mathbf{AB}$ and $\mathbf{BA}$ equal the eigenvalues of $\mathbf{A}$ times the eigenvalues of $\mathbf{B}$.

(B) $\mathbf{A}$ and $\mathbf{B}$ must be similar for the eigenvalues of $\mathbf{AB}$ to be equal to the eigenvalues of $\mathbf{BA}$.

(C) $\mathbf{AB}$ and $\mathbf{BA}$ share the same set of eigenvectors.

(D) Eigenvalues of $\mathbf{A} + \mathbf{B}$ equal the eigenvalues of $\mathbf{A}$ plus the eigenvalues of $\mathbf{B}$.

(E) None of the above are true.

八、 Let $\mathbf{P}$ be a (6×6) non-zero orthogonal real-valued projection matrix. Which of the following are always true?

(A) $\text{rank}(\mathbf{P}) = 6$.

(B) Eigenvectors of $\mathbf{P}$ are linearly independent, but not orthogonal.

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(C) $\mathbf{P}$ is not necessarily symmetric.

(D) The set of eigenvalues of matrix $\left(\mathbf{I}_6 - 2\frac{\underline{v}\underline{v}^\top}{\underline{v}^\top\underline{v}}\right)$ is the same for any nonzero vector $\underline{v} \in \mathbb{R}^6$.

(E) None of the above are true.

九、 Which of the following statements are true?

(A) If the columns of a (9×7) matrix form an orthonormal set, then the same is true of its rows.

(B) If the $(m \times n)$ real-valued matrix $\mathbf{A}$ satisfies $\mathbf{A}^\top\mathbf{A} = \mathbf{I}_n$, then for any nonzero vector $\underline{x} \in \mathbb{R}^m$, $\underline{x} - \mathbf{A}\mathbf{A}^\top\underline{x}$ is orthogonal to the column space of $\mathbf{A}$.

(C) If V is a subspace of an inner-product space W, then every element $\underline{w} \in W$ can be expressed as $\underline{w} = \underline{u} + \underline{v}$, where $\underline{v} \in V$ and $\underline{u}$ lies in the orthogonal complement of V.

(D) If $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ spans a real inner-product space V, and if $\underline{u} \in V$ and $\langle \underline{u}, \underline{v}_i \rangle = 0$ for $i = 1, 2, \ldots, n$, then $\underline{u}$ is the all-zero vector.

(E) None of the above are true.

十、 Which of the following statements are true?

(A) A real square matrix $\mathbf{A}$ may satisfy $\underline{z}^\top\mathbf{A}\underline{z} > 0$ for any nonzero real vector $\underline{z}$, without being symmetric.

(B) Every square matrix has an LU decomposition.

(C) If an $(m \times n)$ real matrix $\mathbf{A}$ has linearly dependent columns and $\underline{b} \in \mathbb{R}^m$, then $\underline{b}$ does not have a unique projection onto the column space of $\mathbf{A}$.

(D) For $x + 3y = 1$, $2x - y = 1$, $4x + y = 1$, the normal equations are $21x + 5y = 7$ and $11x + 5y = 3$.

(E) None of the above are true.

十一、 Which of the following statements are true?

(A) The sample space is a set that contains all real numbers.

(B) The probability measure can assign negative values to some events.

(C) Disjoint events are statistically independent.

(D) The probability of the union of a countably infinite number of disjoint events equals the sum of the probability of each individual event.

(E) None of the above are true.

十二、 Consider a random experiment of rolling a fair 6-face dice twice independently. Let X_1 denote the face value of the first roll and X_2 the second roll's face value. Which of the following statements are true?

(A) X_1 is a random variable that maps possible outcomes to real numbers.

(B) $\{X_1 = 4\}$ refers to the event that the first roll has the 4-point face up.

(C) The probability of $\{X_1 = 1\}$ given $\{X_2 = 1\}$ equals 1.

(D) The probability of $\{X_1 + X_2 > 4\}$ equals $\frac{5}{6}$.

(E) None of the above are true.

十三、 For a discrete random variable X, which of the following statements are true?

(A) $\{X = x\}$ represents an event that contains only one possible outcome of the random experiment.

(B) A function of the random variable, say, X^2, defines another random variable.

(C) The probability of tossing a coin n times and observing k times of heads can be described by the binomial PMF: $p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$, where p is the probability that the head appears in each statistically independent toss.

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(D) The PMF of X^2 is given by $p_{X^2}(x) = [p_X(x)]^2$.

(E) None of the above are true.

十四、 Let X be a geometric random variable with PMF $p_X(k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$; then which of the following statements are true?

(A) $\mathbb{E}[X] = p$.

(B) $\mathbb{E}[\frac{1}{X}] = \frac{1}{p}$.

(C) $\mathbb{E}[X|X > 1] = 1 + p$.

(D) $\Pr(X > 1) = p$.

(E) None of the above are true.

十五、 Let X and Y be two discrete random variables. Which of the following statements are true?

(A) If they are statistically independent, then $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$.

(B) If their covariance equals one, then they cannot be statistically independent.

(C) The marginal PMF $p_X(x)$ obtained from the joint PMF $p_{X,Y}(x, y)$ satisfies $\sum_x p_X(x) = 1$.

(D) The conditional expectation of X given $\{Y = y\}$ is a function of y.

(E) None of the above are true.

十六、 Let X and Y be two statistically independent random variables with mean μ_X, μ_Y, and variance σ_X^2, σ_Y^2, respectively. For any $a > (\mu_X + \mu_Y)^2$, which of the following are upper bounds of $\Pr((X+Y)^2 \geq a)$?

(A) $\frac{\sigma_X^2+\sigma_Y^2}{a}$

(B) $\frac{\sigma_X^2+\sigma_Y^2}{a^2}$

(C) $\frac{\sigma_X^2+\sigma_Y^2}{(\sqrt{a}-|\mu_X+\mu_Y|)^2}$

(D) $e^{-a}\mathbb{E}[e^{(X+Y)^2}]$

(E) None of the above are true.

十七、 Let X and Y be two statistically independent continuous random variables with PDFs $f_X(x)$ and $f_Y(y)$, respectively. Suppose that $Z = X + Y$ and $W = X - Y$ with PDFs $f_Z(z)$ and $f_W(w)$ respectively. With $\sup_z f_Z(z)$ being the supremum of $f_Z(z)$ for all $z \in (-\infty, \infty)$, which of the following statements are true?

(A) $\sup_z f_Z(z) \leq \sup_x f_X(x)$

(B) $\sup_w f_W(w) \leq \sup_y f_Y(y)$

(C) Z and W are statistically independent random variables.

(D) The joint PDF of Z and W is $f_{Z,W}(z, w) = \frac{1}{2} f_X(\frac{z+w}{2}) f_Y(\frac{z-w}{2})$.

(E) None of the above are true.

十八、 Let X and Y be joint normal random variables with mean μ_X, μ_Y and variance $\sigma_X^2, \sigma_Y^2 > 0$, respectively. The correlation coefficient of X and Y is ρ. Which of the following statements are true?

(A) Conditioning on $Y = y$, the random variable $X|\{Y = y\}$ is also normal.

(B) The conditional variance of X given Y is $(1-\rho)\sigma_X^2$

(C) If $\rho = 1$, then $Y = \left|\frac{\sigma_Y}{\sigma_X}\right|(X - \mu_X) + \mu_Y$ with probability one.

(D) If $\rho = 0$, then X and Y are uncorrelated, but not necessarily statistically independent.

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(E) None of the above are true.

十九、 Let Y be a Poisson random variable with PMF $p_Y(y) = e^{-\lambda}\frac{\lambda^y}{y!}$, for $y = 0, 1, \ldots$, and for some parameter $\lambda > 0$. Conditioning on $Y = y$, X is a binomial random variable with PMF $p_{X|Y}(x|y) = \binom{y}{x}p^x(1-p)^{y-x}$, for $x = 0, 1, \cdots, y$, where $p \in [0, 1]$ is some constant. Which of the following statements are true?

(A) $\mathbb{E}[X] = p\lambda$.

(B) The variance of random variable X is $p\lambda$.

(C) Let $\hat{X}$ be an estimate of X based on observation $Y = y$. Then the best $\hat{X}$ which minimizes $\mathbb{E}[(X - \hat{X})^2]$ is $\hat{X} = py$.

(D) X is a Poisson random variable with parameter λp.

(E) None of the above are true.

二十、 Let $X_1, X_2, \cdots$ be a sequence of statistically independent and identically distributed random variables with mean μ and variance σ^2. Define $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for integer $n \geq 1$. Which of the following statements are true?

(A) Random variable S_n has variance equal to σ^2.

(B) If $M_{X_i}(s)$ and $M_{S_n}(s)$ are the moment generating functions of X_i, $i = 1, 2, \cdots$, and S_n, respectively, then $M_{S_n}(s) = \prod_{i=1}^{n} M_{X_i}(s)$.

(C) If $n > 10^4$, $S_n \in (\mu - 0.1\sigma, \mu + 0.1\sigma)$ has a probability larger than 0.99.

(D) $\frac{\sqrt{n}}{\sigma}(S_n - \mu)$ converges in distribution to a standard normal random variable as $n \to \infty$.

(E) None of the above are true.