科目：工程數學 D(3006)

校系所組：中央大學電機工程學系(系統與生醫組)
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參考用

1. (8%) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 9 & 12 \\ 0 & a & b \end{bmatrix}$.

   **(a)** (4%) Suppose $a = 0$ and $b = 4$, find $A^{-1}$.

   **(b)** (4%) Find all nonzero values of $a$ and $b$ so that the nullspace of $A$ has infinitely many vectors. Justify your answer.

2. (15%) Are the following true (T) or false (F)? For each of your answers give a brief explanation or a counterexample.

   **(a)** (3%) If $v_1, v_2, v_3$, and $v_4$ are linearly independent vectors in $\mathbf{R}^4$, then so are $v_1, v_2$, and $v_3$.

   **(b)** (3%) If $A$ and $B$ are both $n$ by $n$ matrices, then $(A+B)^2 = A^2 + 2AB + B^2$.

   **(c)** (3%) Suppose $A$ is an $m$ by $n$ matrix with rank $r$, and $b$ is a column vector with $m$ entries. If $m > r$ and $n > r$, then the equation $Ax = b$ has infinitely many solutions for some $b$ and exactly one solution for other $b$.

   **(d)** (3%) The maximum number of distinct entries of a square $n$ by $n$ symmetric matrix is $n^2$.

   **(e)** (3%) If $A$ is similar to $B$, then $A^3$ is similar to $B^3$.

3. (10%) Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$.

   **(a)** (4%) Find an orthonormal basis for the column space of $A$ using the Gram-Schmidt procedure.

   **(b)** (3%) Let $P_1$ be the projection matrix that projects $\mathbf{R}^4$ onto the subspace spanned by the first column of $A$, and $P_2$ be the projection matrix that projects $\mathbf{R}^4$ onto the column space of $A$. Find the product $P_1P_2$.

   **(c)** (3%) Prove that all projection matrices are symmetric. Please explain each of your steps in detail.

4. (4%) Consider the basis $\mathbf{S} = \{v_1, v_2, v_3\}$ for $\mathbf{R}^3$, where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, and $v_3 = (1,0,0)$. Suppose the linear transformation $\mathrm{T} : \mathbf{R}^3 \to \mathbf{R}^3$ is such that

$$\mathrm{T}(v_1) = (1,2,3),\ \mathrm{T}(v_2) = (4,5,6),\ \mathrm{T}(v_3) = (7,8,9).$$

   Find a formula for $\mathrm{T}(x)$, where $x = (x_1, x_2, x_3)$, and use that formula to find $\mathrm{T}(v)$ for $v = (1,2,1)$.

5. (13%)

   **(a)** (2%) Find the eigenvalues of $A = \begin{bmatrix} 1 & -12 \\ -12 & -6 \end{bmatrix}$.

   **(b)** (2%) Find a matrix $P$ that diagonalizes $A$ in **(a)**.

   **(c)** (3%) Show that $B = \begin{bmatrix} -11 & -12 \\ -7 & 6 \end{bmatrix}$ is similar to $A$ in **(a)**.

   **(d)** (3%) Let $C = \begin{bmatrix} a & 6 & 0 \\ 6 & a & 8 \\ 0 & 8 & a \end{bmatrix}$. Find the condition(s) on the number $a$ so that the matrix $C$ has only positive eigenvalues.

   **(e)** (3%) Show that the matrix $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is not diagonalizable.

注意：背面有試題

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6. (12%) Consider the following initial-value problem

$$y'' + ay' + by = f(x), \quad y(0) = c, \ y'(0) = d$$

where $a, b, c$, and $d$ are (real) constants.

**(a)** (4%) It is known that the solution is $y(x) = \sin 2x$ when $f(x) = -3\sin 2x$. Determine the values of the constants $a, b, c$, and $d$.

**(b)** (8%) Find the solution $y(x)$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, when $f(x) = \tan x$.

7. (13%)

**(a)** (6%) Determine the Laplace transform $F(s)$ of the function $f$ defined as follows

$$f(t) = \begin{cases} 1, & 0 \le t < 2; \\ -3, & 2 \le t < 3; \\ t^2, & t \ge 3. \end{cases}$$

**(b)** (7%) Let $h$ be the convolution of the functions $w$ and $f$, *i.e.*,

$$h(t) = \int_{-\infty}^{\infty} w(t-\tau) f(\tau) d\tau$$

where $w : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ are defined as

$$w(t) = \begin{cases} 1, & 0 \le t < 1; \\ 0, & \text{otherwise.} \end{cases} \qquad f(t) = \begin{cases} 2t, & 0 \le t < 1; \\ 4-2t, & 1 \le t < 2; \\ 0, & \text{otherwise.} \end{cases}$$

Compute $h(t)$ for $1 \le t < 2$.

8. (12%) Consider the following differential equation

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 - 2y_2 \end{aligned}$$

**(a)** (8%) Determine the set of all solutions to the above differential equation.

**(b)** (4%) Suppose $y_1(0) = 0$ and $y_2(1) = 1$. Determine $y_1(t)$ and $y_2(t)$ for $t \ge 0$.

9. (13%) Consider the differential equation

$$\begin{aligned} \frac{dy_1}{dt} &= -y_2 - y_1(y_1^2 + y_2^2) \\ \frac{dy_2}{dt} &= y_1 - y_2(y_1^2 + y_2^2) \end{aligned}$$

**(a)** (6%) Convert the equations into a set of differential equations in $r$ and $\theta$, where $(y_1, y_2)$ and $(r, \theta)$ are related by $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$.

**(b)** (7%) Determine $y_1(t)$ and $y_2(t)$ for $t \ge 0$, given the conditions $y_1(0) = 0, y_2(0) = 1$.