1. True or False? [21 points] Please briefly justify your answer. Answers without correct justification will get zero point.

(1-1) [3 points] $n \le 3^{\frac{n}{3}}$ for every non-negative integer, n.

(1-2) [3 points]) In a group of five people, where each two are either friends or enemies, there must be either three mutual friends, or three mutual enemies.

(1-3) [3 points] If a is an integer and m is a positive integer, then $a^{m-1} \equiv 1 \pmod{m}$.

(1-4) [3 points] If all the edge weights on a graph are distinct, the minimum spanning tree is unique.

(1-5) [3 points] Let p be a positive integer, and $s_1, s_2, \cdots, s_n$ an be integers. If p divides $s_1 s_2 \cdots s_n$, then p divides s_i for some i.

(1-6) [3 points] Let G be a graph with v vertices and e edges. Let m be the minimum degree of the vertices of G. We have $2e/v \ge m$.

(1-7) [3 points] Given a set of $n+2$ positive integers, none exceeding $2n$, there is at least two integers in this set that divides another integer in the set.

2. Multiple Choices (select one or more answer choices). [10 points] Each option is worth of 1 point. However, a wrong choice gets -1 point penalty. For example, if the correct answer is (a)(b)(c) and your answer is (d)(e), you get -5 points. If your answer is (a)(b), you get 3 points. If you left it blank, you get zero point.

(2-1) [5 points] Select the true statement below. Suppose you do not know what my favorite number is (but you do know that 31 is a prime).

(a) If 31 is prime, then 31 is my favorite number.
(b) If 31 is my favorite number, then 31 is prime.
(c) If 31 is not prime, then 31 is my favorite number.
(d) 31 is my favorite number or 31 is prime.
(e) 31 is my favorite number or 31 is not my favorite number.

(2-2) [5 points] Let A, B, and C be sets, and let f: $B \to C$ and g: $A \to B$ be functions. Let h: $A \to C$ be the composition, f∘g, that is, h(x)::== f(g(x)) for x∈ A.

(a) If h is surjective, then f must be surjective.
(b) If h is surjective, then g must be surjective.
(c) If h is injective, then f must be injective.
(d) If h is injective and f is total, then g must be injective.
(e) If h is injective and g is total, then f must be injective.

[Hints] The function is injective, or one-to-one, if each element of the codomain is mapped to by at most one element of the domain. A function is surjective (onto) if each element of the codomain is mapped to by at least one element of the domain. A function is total if it is defined for all possible input values.

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3. Take control of your own destiny! [13 points]
The house price is too high now in Taiwan. Therefore, Dr. X decides to play a game on TV. The TV show uses a round-robin tournament for the audition. Every two distinct players play against each other just once. For a round-robin tournament with no tied games, a record of who beat whom can be described with a tournament digraph, where the vertices correspond to players and there is an edge $x \to y$ iff x beat y in their game. Moreover, a ranking is a path that includes all the players. In other words, in a ranking, each player won the game against the next lowest ranked player, but may very well have lost their games against much lower ranked players-whoever does the ranking may have a lot of room to play favorites.
(3-1) [4 points] Give an example of a tournament digraph with more than one ranking.
(3-2) [4 points] Try to prove that if a tournament digraph is a directed acyclic graph, then it has **at most** one ranking.
(3-3) [5 points] Prove that every finite tournament digraph has a ranking.

4. Win a prize!! [13 points] After winning the audition, Dr. X finally gets the chance to win a house in the TV show. The game in the TV show is similar to the famous game of Monty Hall's game (see the hint below). The player still picks one of three doors, with a prize of house randomly placed behind one door and goats behind the other two. However, instead of always opening a door to reveal a goat, the host now instructs Dr. YY to randomly open one of the two doors that the Dr. X has not picked. This means Dr. YY may reveal a goat, or Dr. YY may reveal the prize. If Dr. YY reveals the prize, then the entire game is restarted, that is, the prize is again randomly placed behind some door, the player again picks a door, and so on until Dr. YY finally picks a door with a goat behind it. Then the Dr. X can choose to stick with her original choice of door or switch to the other unopened door. Dr. X wins if the prize is behind the door she finally chooses. Should Dr. X change? Calculate the probability that changing the door will win the price to justify the answer.

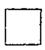
[Hints] We describe the original Monty Hall's game here. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a house; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

5. Dreams made into reality [14 points]

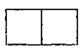
(5-1) [6 points] After winning the house, Dr. X puts her favorite unit square on the floor as shown in Table 1(a). It is easy to find that the length of the periphery of the square is 4, which is an even number. When she puts another square adjacent to the first as shown in Table 1(b). The length of the periphery of the two squares is now 6, which is also an even number. Use induction on the number of squares to prove that the length of the periphery is always even.

Table 1: An illustrative example for Latin square.

(a)



(b)



(5-2) [8 points] Now Dr. X finds a geometric object known as the Koch snowflake to decorate the roof. Koch snowflake is defined by a sequence of polygons S_0, S_1 recursively. We start with S_0, which is an equilateral triangle with unit sides. We then construct S_{n+1} by removing the middle third of each edge of S_n and replacing it with two line segments of the same length. Figure 1 illustrates the construction process.

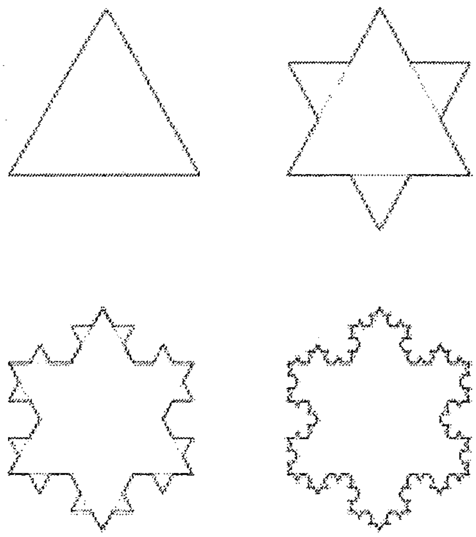


Figure 1: An illustrative example of Koch snowflake.

Let a_n be the area of S_n. For example, a_0 is just the area of the unit equilateral triangle which by elementary geometry is $\frac{\sqrt{3}}{4}$. Prove by induction that for $n \geq 0$, the area of the n-th snowflake is given by

$$a_n = a_0(\frac{8}{5} - \frac{3}{5}(\frac{4}{9})^n). \quad (1)$$

6. Finalize the decoration. [12 points] Dr. X now wants to paint something on the unit square. She finds an interesting pattern called Latin square, which is an $n \times n$ array whose entries are the number $1, \cdots, n$. These entries satisfy two constraints: every row contains all n integers in some order, and also every column contains all n integers in some order. For example, Table 2(a) is a 3×3 Latin square.

Table 2: An illustrative example for Latin square.

(a)

2	1	3
1	3	2
3	2	1

(b)

3	2	1	4	5
1	4	2	5	3
2	5	3	1	4

(6-1) [2 points] Fill in the last two rows of Table 2(b) to extend this "Latin rectangle" to a complete Latin square.
(6-2) [3 points] Show that filling in a row of an $n \times n$ Latin rectangle is equivalent to finding a matching in some $2n$-vertex bipartite graph.
(6-3) [7 points] Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.

7. Divide-and-conquer. [7 points] Prove or disprove that if p is a prime greater than 3, then $p^2 - 1$ is divisible by 24.

8. Details are important, be observant. [10 points]
(a) [5 points] Define a function $f(n)$ such that $f = \Theta(n^2)$ and NOT $f \sim n^2$.
(b) [5 points] Define a function $g(n)$ such that $g = O(n^2)$, $g \neq \Theta(n^2)$, $g \neq o(n^2)$, and $n = O(g)$.

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