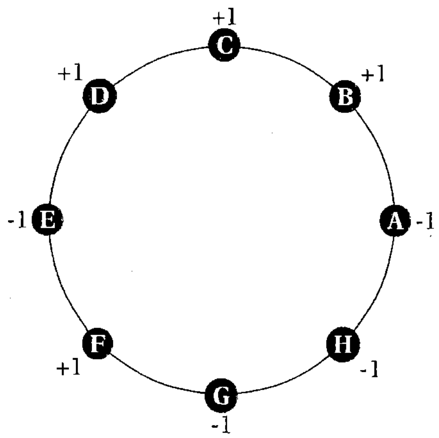
1. **Let's play a game: Circle Game [12 points]** We have a circle with $2n$ arbitrarily-chosen points for some natural number $n \geq 1$. Moreover, among the $2n$ points, n are labeled $+1$, while the remaining n are labeled -1. Take the figure below as an example, which contains eight points with four nodes are labeled $+1$ and four are labeled -1.

Here's a game you can play. Pick one of the $2n$ points as your starting point, then move clockwise around the circle. You lose the game if at any point on you pass through more -1 points than $+1$ points. You win the game if you get all the way back to your starting point without losing. For example, if you start at point C, the game would go like this:

Start at C: +1.
Pass through B: +2.
Pass through A: +1.
Pass through H: 0.
Pass through G: -1. *(You lose.)*



If you started at point G, the game would go like this:

Start at G: -1 *(You lose.)*

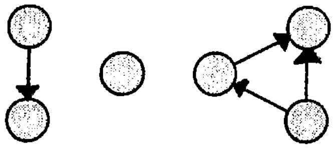
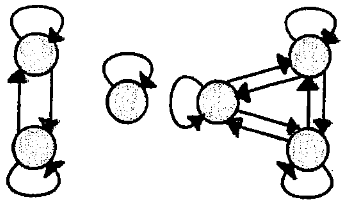
However, if you started at point F, the game would go like this:

Start at F: +1.
Pass through E: 0.
Pass through D: +1.
Pass through C: +2.
Pass through B: +3.
Pass through A: +2.
Pass through H: +1.
Pass through G: +0.
Return to F. *(You win!)*

No matter which n points are labeled $+1$ and which n points are labeled -1, there is always at least one point you can start at to win the game. Prove, by induction, that the above fact is true for any $n \geq 1$.

2. Brute force cannot solve everything. [8 points] Use Fermat's Little Theorem to compute 3^{302} mod 5.

3. Define the relation. **[9 points]** Below is a collection of pictures of binary relations.

Relation 1	***Relation 2***	***Relation 3***
		*(This is intentionally blank.)*

Fill in the table below. No justification is necessary. However, you only get 3 points for a relation if all the answers for the relation are correct.

	Reflexive	Symmetric	Transitive	Irreflexive	Asymmetric	Equiv. Rel.	Strict Order
Relation 1							
Relation 2							
Relation 3							

4. Let's exchange! **[12 points]** Let T_1 and T_2 be spanning trees of G with $T_1 \neq T_2$. Prove that there exists an edge e_1, where $e_1 \in E(T_1) \wedge e_1 \notin E(T_2)$ and an edge e_2, where $e_2 \in E(T_2) \wedge e_2 \notin E(T_1)$ so that both $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are spanning trees.

5. Opportunity knocks but once. It's now or never. **[10 points]** A robot moves on the two-dimensional integer grid. It starts out at (0,0) and is allowed to move in any of these four ways: [1] (+2,-1): right 2, down 1, [2] (-2,+1): left 2, up 1, [3] (+1,+3), an [4] (-1,-3). Prove that this robot can never reach (1,1).

6. Coloring the graph, coloring your life. **[12 points]** Let the vertices of a graph G be the integers $1, 2, \cdots, 40$. The numbers $i \neq j$ are connected if they are not relatively prime numbers. Find the chromatic number of G, i.e., the minimum number of colors for coloring the nodes so that two nodes connected by an edge are with different colors.

7. Clique, clique. **[12 points]** A k-clique is a graph with k nodes where each node is connected to the $k-1$ other nodes in the graph. Now, suppose that you take a k-clique and color each **edge** either red or blue. Prove the following result by induction: if the k-clique contains an odd-length cycle made only of blue edges, then it must contain a cycle of length three with an odd number of blue edges (that is, a cycle of length three with exactly one blue edge or exactly three blue edges.)

8. Master is not enough (go for PhD?). [19 points]

(a) [5 points] Give an example of recurrences that is in the form of T(n) = aT(n/b)+f(n) but cannot be solved with Master theorem.

(b) [14 points] Now, let's try to use the recurrence tree method to solve the time complexity of recurrence $T(n) = \sqrt{n}T(\sqrt{n}) + n$ in the θ-notation.

9. Wild Guess. [6 points] How many 0s are at the end of 20! when written in octal (base-8)? Briefly explain your answer.