國立中央大學九十一學年度碩士班研究生入學試題卷

所別: 物理學系 不分組 科目: 應用數學 共 2 頁 第 1 頁

Suggestions: Do the "easy" ones first but do try every problem. Partial credit will be given for good attempts.

1. (a) (10 pts) Please find the *eigenvalues* and *eigenvectors* of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) (5 pts) Please expand and simplify: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$.

(c) (5 pts) Please calculate $\nabla_r (r^{-2}\hat{\mathbf{r}})$. Here $\hat{\mathbf{r}}$ is the unit vector $\mathbf{r}/r$ where $\mathbf{r}$ is the usual vector with rectangular components (x, y, z) and $r = |\mathbf{r}|$.

(d) (10 pts) For $\mathbf{F} = (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})/(x^2 + y^2)$ please calculate

$$\nabla \times \mathbf{F} \qquad \text{and} \qquad \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the unit circle $x^2 + y^2 = 1$ in the plane $z = 3$.

2. (a) (5 pts) Please find the Laurent expansion of $f(z) = [z(z-1)]^{-1}$ around $z_0 = 1$.

(b). (10 pts) Please evaluate the following integral using $z = e^{i\theta}$

$$I_1 = \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}.$$

(c). (10 pts) Please find the value of the real definite integral

$$\int_{-\infty}^{+\infty} \frac{dx}{1 + x^4}$$

by using the residue theorem (with a suitable contour) to evaluate $\oint_c f(z)dz$ with $f(z) = 1/(1 + z^4)$.

3. The Helmholtz wave equation in cylindrical coordinates (ρ, φ, z) is given by

$$\nabla^2\psi + k^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0. \qquad (*)$$

The solutions to this partial differential equation (PDE) can be found by the technique of *separation of variables*.

(a) (10 pts) Assume $\psi(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$. Substitute into equation (*) to obtain 3 separate second order *ordinary* differential equations (ODEs) for $R(\rho)$, $\Phi(\varphi)$, $Z(z)$.

(b) (5 pts) The second order equations for $\Phi(\varphi)$, $Z(z)$ are very simple. Please give two linearly independent solutions for each of these equations.

Solutions to the second order ODE for $R(\rho)$ can be found by the series technique (Frobenius). Assume a series of the form

$$R = \rho^k \sum_{n=0}^{\infty} a_n \rho^n,$$

substitute into the ODE and collect coefficients of like powers of ρ. The coefficient of the lowest power of ρ yields the *indicial* equation, which determines the possible values of k. The vanishing of the other coefficients yields a *recurrence relation* connecting the a_n's.

(c) (10 pts) Please find this recurrence relation and indicial equation.

4. (a) (10 pts) Please find a *Fourier series* expansion for the periodic sawtooth function: $y = x$ for $-L/2 < x < L/2$ and $y(x + L) = y(x)$.

(b) (10 pts) What is the *fourier transform* of the 2nd order linear ODE (with constant coefficients)

$$y'' + ay' + by = F_0 \cos\omega_0 t,$$

where F_0 and ω_0 are constants?