國立中央大學107學年度碩士班考試入學試題

所別： 天文研究所 碩士班 不分組(一般生)　　　　共2頁 第1頁

科目： 應用數學

本科考試禁用計算器　　　　*請在答案卷(卡)內作答

1. **(Total 15%)** The Gamma function is defined as

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t} dt$$

(i) (10%) Prove $\Gamma(x+1) = x\Gamma(x)$

(ii) (5%) Calculate $\Gamma(\frac{5}{2})$

2. **(Total 20%)** If $\vec{r}$ is the position vector and $\vec{a}$ is a constant vector in three dimensional space, that is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ where $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors of x, y, z axis respectively.

(i) (10%) What kind of surface is if $(\vec{r} - \vec{a}) \cdot \vec{r} = 0$ and what is its surface area?

(ii) (10%) Calculate $|\nabla[(\vec{r} - \vec{a}) \cdot \vec{r}]|$ on the surface where ∇ is gradient operator.

3. **(Total 15%)** Show that $\tan(x+iy) = \dfrac{\sin(2x) + i\sinh(2y)}{\cos(2x) + \cosh(2y)}$

4. **(Total 20%)** The expected value of a probability distribution $P(x)$ is defined as $\langle f(x) \rangle = \int f(x)P(x)dx$. For the probability distribution $P(x) = A\frac{1}{\sqrt{1-x^2}}$ where $-1 < x < 1$ and A is the normalization constant to make $\int P(x)dx=1$, find

(i) (5%) The normalization constant A

(ii) (5%) Mean value $\mu \equiv \langle x \rangle$

(iii) (5%) Variance $\sigma^2 \equiv \langle (x - \langle x \rangle)^2 \rangle$

(iv) (5%) The probability between $\mu - \sigma$ and $\mu + \sigma$ where $\sigma = \sqrt{\sigma^2}$

參考用

注意:背面有試題

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5. **(Total 15%)** The solution of the differential equation

$\left(1-x^2\right)y''-2xy'+l\left(l+1\right)y=0$ can be written as a power series as $y=\sum_{n=0}^{\infty}c_n x^n$

(i) (10%) Show that

$$c_{n+2}=-\frac{(l-n)(l+n+1)}{(n+2)(n+1)}c_n$$

(ii) (5%) Find the following solutions:

(a) $c_0=1, c_1=0, l=0$; (b) $c_0=1, c_1=0, l=2$; (c) $c_0=0, c_1=1, l=1$;

(d) $c_0=0, c_1=1, l=3$

6. **(Total 15%)** A is an $n\times n$ Hermitian matrix with orthonormal eigenvectors $|x_i\rangle$ and the corresponding real eigenvalues $\lambda_1\le\lambda_2\le\lambda_3\cdots\le\lambda_n$. Show that a unit vector $|y\rangle$

$$\lambda_1\le\langle y|A|y\rangle\le\lambda_n$$

Hint: You may consider $|x_i\rangle$ as a column matrix as $\begin{pmatrix}x_{i1}\\x_{i2}\\\vdots\\x_{in}\end{pmatrix}$ and its adjoint

$\langle x_i|=\left(|x_i\rangle\right)^\dagger$ as a raw matrix as $\left(x_{i1}^*, x_{i2}^*, \cdots, x_{in}^*\right)$.

參考用

注意:背面有試題