國立中央大學101學年度碩士班考試入學試題卷

所別：天文研究所碩士班 不分組(一般生)　　科目：應用數學　　共＿一＿頁　第＿一＿頁
天文研究所碩士班 不分組(在職生)

本科考試禁用計算器　　　　＊請在試卷答案卷（卡）內作答

(1) (20 points)
The position vector of a point in two-dimensional space is given by $\vec{r}(t)$ where t is time. It can be expressed in Cartesian coordinates and polar coordinates $\vec{r} = x\hat{e}_x + y\hat{e}_y = r\hat{e}_r$, where $x = r\cos\phi$ and $y = r\sin\phi$.

(a) (5 points) Write down the orthogonal basis vectors of the polar coordinate system $\hat{e}_r$ and $\hat{e}_\phi$ in Cartesian coordinate system (i.e., in terms of the Cartesian basis vectors $\hat{e}_x$ and $\hat{e}_y$).

(b) (15 points) Derive the velocity $\vec{v} = d\vec{r}/dt$ and the acceleration $\vec{a} = d^2\vec{r}/dt^2$ in polar coordinate system (i.e., in terms of (r, ϕ), their derivatives, and $\hat{e}_r$, $\hat{e}_\phi$).

(2) (10 points)
Plot the following two plane curves in Cartesian $x-y$ plane. The two curves are expressed in parametric form. Simple labelling of the axes is required.

(a) (5 points) $x = -\sin\theta$ and $y = 1 - \cos\theta$, $0 \le \theta \le 2\pi$.

(b) (5 points) $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$, $0 \le \theta \le 2\pi$.

(3) (15 points)
A closed plane curve is given by $(x, y) = (a\cos\theta, b\sin\theta)$, where θ goes from 0 to 2π, and a and b are constants. Derive the area enclosed by the curve.

(4) (10 points)
$z = x + i\,y$ is a complex number ($i = \sqrt{-1}$). $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function of z. Here x, y, u and v are real.

(a) (5 points) Write down the Cauchy-Riemann equations, and show that u and v satisfy the two-dimensional Laplace equation $\nabla^2 u = \nabla^2 v = 0$.

(b) (5 points) If $z^2 = a + i\,b$ (a and b are real), find x and y in terms of a and b.

(5) (15 points)
Find the eigenvalues (in terms of β) and the corresponding normalized eigenvectors of the matrix

$$\mathcal{M} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

(6) (15 points)
Solve $x(t)$ and $y(t)$ of the following set of ordinary differential equations

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = -2x - y, \qquad \text{and} \qquad x(t=0) = 1, \quad y(t=0) = -1.$$

(7) (15 points)

(a) (5 points) Show that $f(x, y) = F(x - y)$ ($F(\xi)$ is an arbitrary function of ξ) is the general solution of the first order partial differential equation,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0.$$

(b) (10 points) Find the general solution of

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = (x + y)\,g.$$

Hint: Try $g(x, y) = G_1(x - y)G_2(x + y)$.