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(5%)　1.　(a) Ordinary three-dimensional space may be described with the familiar cartesian coordinates $X - Y$. Now, we consider a second set of cartesian coordinates $X' - Y'$ whose origin coincides with that of the first set but whose orientation is different from the first set with the angle $\phi$ along the $Z$ axis (Fig. 1). In general, these two representations can be connected with:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

Please explore all the $a_{ij}$ in term of the rotation angle $\phi$.

(5%)　　(b) Calculate the eigenvalues and the corresponding eigenvectors of the matrix $A \equiv (a_{ij})$ in Eq.(1).

(5%)　　(c) As considering the successive powers of the complex number $i^n$, $n = 0, 1, 2, 3, \cdots$, it may be interpreted as successive 90° rotation in the complex plane. Express a complex number $Z = R + iI$ in the matrix representation. ($i = \sqrt{-1}$)
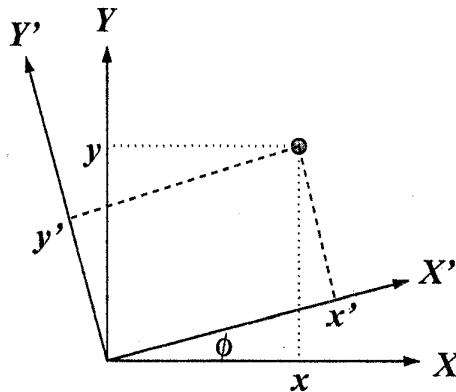


Figure 1:

2. Derive the binomial series expansions for

(5%)　　(a) $\dfrac{x}{1-x} = x\dfrac{1}{1-x} = \sum_{n=1}^{\infty} x^n$ , and

$\dfrac{x}{x-1} = \dfrac{1}{1-x^{-1}} = \sum_{n=0}^{\infty} x^{-n}$

(5%)　　(b) If adding these two series, we will obtain $\sum_{n=-\infty}^{\infty} x^n = \dfrac{x}{1-x} + \dfrac{x}{x-1} = 0$. What's wrong?

3. With the calculus of residues, show that

(5%)　　(a) $\displaystyle\int_0^\pi \dfrac{x^a}{(x+1)^2}dx$

(5%)　　(b) $\displaystyle\int_0^\pi \dfrac{d\theta}{(a+\cos\theta)^2}$, 　　$a > 1$

(5%)　　(c) $\displaystyle\int_{-\infty}^\infty \dfrac{\sin x}{x}dx$

(10%)　4. For a simple lens of focal length $f$ the object distance $p$ and the image distance $q$ are related by $1/p + 1/q = 1/f$. Find the minimum object-image distance $(p + q)$ for fixed $f$ by using Lagrangian multipliers. Assume real object and image ($p$ and $q$ both positive).

注意：背面有試題

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5. (10%)

請解 Partial Equation $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

6. (10%)

請証明下列方程式為 exact，並解其方程式。

$(2x + e^y)dx + (xe^y)dy = 0$

7. (10%)

請解　$y' = y \tan 2x, \quad y(0) = 2$

8. (10%)

請解　$y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0,$

9. (10%)

請解　$y'' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 3\sqrt{3}$