國立中央大學101學年度碩士班考試入學試題卷

所別：光電科學與工程學系碩士班 不分組(一般生)　科目：工程數學　共 2 頁 第 1 頁
光電科學與工程學系碩士班 不分組(在職生)

本科考試可使用計算器，廠牌、功能不拘　　*請在試卷答案卷（卡）內作答

1. Find the particular solutions of the following equations:

5% (a) $x' + 2x = 3t.$

5% (b) $x' + 2x = 3e^{-t}$

5% (c) $tx' + 2x = 3e^{-t}$

5% (d) $0.25x'' + x' + 2x = 3e^{-t}\sin 2t$

2. Super model Chi-Ling Lin is on a diet. Her basic metabolism consumes 1,200 calories per day. Her exercise consumes 15 calories per hour per kilogram of body mass. She takes 1 hour 40 minutes per day. Food intake provides 800 calories per meal. She has three meals per day. Caloric intake that is not consumed by basic metabolism or exercise is converted into fat; calories needed for basic metabolism or exercise in excess of caloric intake are obtained from the fat store. Body fat stores or releases calories at a rate of 7,500 calories per kilogram of fat. Assume that the conversion of calories to fat is perfect.

10% (a) Denote by $W(t)$ the mass, in kilograms, of Chi-Ling on day t of this diet regimen. Write a differential equation for $W(t)$ using the information given.

5% (b) Find the solution of $W(t)$. Sketch some solutions.

5% (c) If Chi-Ling sticks religiously to this diet, what is his ultimate mass?

10% 3. Invert the Laplace transform $f(s) = 1/[(s+a)(s+b)]$, $a \neq b$, by the convolution theorem.

4. Consider a finite wave train defined by:

$$f(t) = \begin{cases} \sin\omega_0 t & |t| < \dfrac{N\pi}{\omega_0} \\ 0 & |t| > \dfrac{N\pi}{\omega_0} \end{cases}$$

5% (a) Perform appropriate Fourier transform of the finite wave train and plot the function out. Note that you need to specify the value of the maximum amplitude and the location where it occurs.

5% (b) Evaluate the quality factor **Q** ($\equiv \frac{\Delta\omega}{\omega_0} = \frac{\omega_0 - \omega}{\omega_0}$) and describe how it varies with N.

5% 5. With φ a scalar function, show that $(r \times \nabla)\cdot(r \times \nabla)\varphi = r^2\nabla^2\varphi - r^2\frac{\partial^2\varphi}{\partial r^2} - 2r\frac{\partial\varphi}{\partial r}$

5% 6. Prove that $\oint u\nabla v \cdot d\lambda = \int (\nabla u) \times (\nabla v) \cdot d\sigma$

7. In the beginning of the 1930, T. Smith formulated a rather interesting way of handling the ray-tracing equations for light. The propagation combines with the transfer steps and the refraction steps.

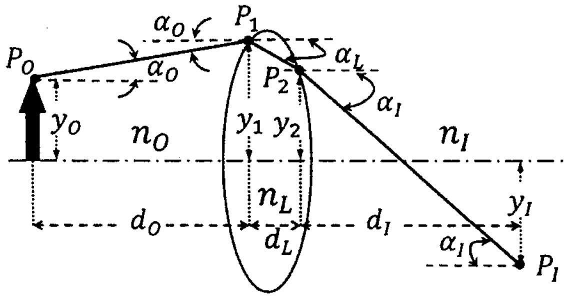

Fig. 1:

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For any ray, saying from P_0 via P_1, P_2, to P_I (refer to Fig. 1), the light ray traveling in a direction that makes angles α_O, α_L, and α_I with the optical axis and the heights y_O, y_1, y_2, y_I perpendicularly off the optical axis, respectively. n_O, n_L, and n_I are the indices of refraction of the medium, respectively. Assume all the direction angles are very small such as $\tan\alpha \cong \sin\alpha \cong \alpha$. The light ray vectors are denoted as $[n_k\alpha_k, y_k]^T$.

5% (a) In the transfer steps (from P_O to P_1, from P_1 to P_2, and from P_2 to P_I), the ray propagates in the same medium ($n_f = n_i = n$) and keeps the direction angles unchanged. Then, we have

$$\begin{cases} \alpha_f = \alpha_i \\ y_f = d \cdot \alpha_i + y_i \end{cases},$$

where the subscripts *i* and *f* denote the initial position and the final position, respectively. *d* is the distance between the initial and the final points. Accordingly, it is easy to construct a *transfer matrix* $\boldsymbol{T_{fi}}$

$$\boldsymbol{T_{fi}} \equiv \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix},$$

such that the ray vectors propagate with $\begin{bmatrix} n_f \cdot \alpha_f \\ y_f \end{bmatrix} = \boldsymbol{T_{fi}} \begin{bmatrix} n_i \cdot \alpha_i \\ y_i \end{bmatrix}$. Find all the $t_{ij}'s$.

5% (b) In the refraction steps (from media n_O to media n_L at P_1 and from media n_L to media n_I at P_2), the ray keep at the same height but changes the direction angles. Then, we have

$$\begin{cases} n_f\alpha_f &= n_i\alpha_i - y_i \cdot (n_f - n_i)/R \\ y_f &= y_i \end{cases},$$

where the subscripts *i* and *f* denote the initial position and the final position, respectively. *R* is the radius of the curvature of the interface. Accordingly, it is easy to construct a *refraction matrix* $\boldsymbol{R_{fi}}$

$$\boldsymbol{R_{fi}} \equiv \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix},$$

such that the ray vectors propagate with $\begin{bmatrix} n_f \cdot \alpha_f \\ y_f \end{bmatrix} = \boldsymbol{R_{fi}} \begin{bmatrix} n_i \cdot \alpha_i \\ y_i \end{bmatrix}$. Find all the $r_{ij}'s$.

5% (c) All the ray propagation form P_O to P_I becomes $\begin{bmatrix} n_I \cdot \alpha_I \\ y_I \end{bmatrix} = \boldsymbol{P} \begin{bmatrix} n_O \cdot \alpha_O \\ y_O \end{bmatrix}$, where $\boldsymbol{P} \equiv \boldsymbol{T_{I2}R_{IL}T_{21}R_{LO}T_{1O}}$.

Find all the elements of the system matrix $\boldsymbol{P}$.

(Note that takes the radii of the curvature at P_1 and P_2 as $+R$ and $-R$, respectively.)

5% (d) For the case of thin lens ($d_L \approx 0$), find out the special distance $d_O = \boldsymbol{f}$ such that all the rays emitted from the optical axis ($y_O = 0, \forall\alpha_O$) will parallel to the optical axis ($\alpha_I = 0$) in the media n_I.

8. Evaluate the following integrals.

5% (a) $\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta} d\theta$

5% (b) $\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} d\theta$

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