國立中央大學九十學年度碩士班研究生入學試題卷

所別： 統計研究所 不分組 科目： 數理統計 共 1 頁 第 1 頁

[15%] 1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli random variables with parameter θ, $0 < \theta < 1$. Find the conditional distribution of X_1 given the value of $T(X_1, \ldots, X_n) = X_1 + X_2 + \ldots + X_n$. What can you conclude on $T(X_1, \ldots, X_n)$? Give your reason.

[20%] 2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution over $(\theta - 1/2, \theta + 1/2)$, for some $\theta \in R$.

a) Find the maximum likelihood estimator (MLE) of θ.

b) Show that $\hat{\theta} = (X_{(1)} + X_{(n)})/2$ is an unbiased estimator of θ, where $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$ and $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$. Is it an MLE?

[10%] 4. Let X be a single observation from the following distribution

x	1	2	3	4
$Pr(X = x)$	θ	3θ	$0.8 - 3\theta$	$0.2 - \theta$

where $\theta \in [0, 0.1]$. Find the MLE of θ.

[20%] 5. Let (X, Y) be uniformly distributed on the unit disk $x^2 + y^2 \le 1$.

a) Show that $E(XY) = [E(X)][E(Y)]$. Are X and Y independent? Why?

b) Find the conditional expectation of X given $Y = 1/\sqrt{2}$.

[10%] 6. Let $X_1, X_2, \ldots, X_n$ be a random sample with pdf

$$f(x|\mu) = \exp\{-(x - \mu)\}, \ \text{, where } x > \mu$$

and $-\infty < \mu < \infty$ is an unknown parameter. Let $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ be their order statistics and $\bar{X} = \sum_{i=1}^{n} X_i/n$. Determine which of the followings are sufficient statistics for μ. Give brief reasons.

(A) $\bar{X} - \mu$ (B) $(X_1, X_2, \ldots, X_n)$ (C) $X_{(n)} - X_{(1)}$

(D) $X_n - X_1$ (E) $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ (F) $\sigma^2 = \sum_{i=1}^{n}(X_i - \mu)^2/n$.

[15%] 3. Let $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where ϵ_i are i.i.d. $N(0, \sigma^2)$ random variables, x_i are given constants, for $i = 1, \ldots, n$; β_0, β_1 and σ^2 are unknown paramters. Note that the MLE's of β_0 and β_1 are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \text{ and } \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}, \text{ respectively.}$$

Find a $(1 - \alpha)100\%$ confidence interval for β_1.

[10%] 7. It is desired to test $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$, based on observing X which assumes the values 1, 2, or 3 with probabilities

θ \ X	1	2	3
0	0.09	0.01	0.9
1	0.01	0.89	0.1

Find the $\alpha = .1$ level most powerful test and its Type I and Type II error probabilities.