國立中央大學八十五學年度碩士班研究生入學試題卷

所別：統計研究所　甲組　　科目：甲基礎數學　　

1. [10 %] Let $\{f_n\}$ be a sequence of functions on a finite interval $[a,b]$ such that the derivative $f_n'(x)$ exists for every $x \in [a,b]$ and every positive integer n. Show that if $f_n(x_0)$ converges for some $x_0 \in [a,b]$ and f_n' converges uniformly on $[a,b]$ then f_n converges uniformly on $[a,b]$.

2. [10 %] If $A = (a_{ij})$ is positive definite and $A^{-1} = (a^{ij})$ then $a^{11} \geq 1/a_{11}$.

3. [15 %] Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\ e^{-t^2/2} dt$.

   a) Prove that

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{1!2\cdot 3} + \frac{x^5}{2!4\cdot 5} - \cdots + (-1)^k \frac{x^{2k+1}}{k!2^k(2k+1)} + \cdots\right).$$

   b) How many terms one needs to evaluate $\Phi(x)$ with 4 correct digits (i.e. with error less than 10^{-4})?

4. [15 %] Let A, B be $n \times n$ matrices such that there exits a nonsingular matrix P (not necessarily orthogonal) satisfying $PAP^{-1} = B$. Show that the eigenvalues of A and B are the same and hence $|A| = |B|$, trace(A)= trace(B).

5. [10 %] Find orthonormal vectors that span the same subspace as

$$V_1 = (1\ 1\ 1\ 1), \qquad V_2 = (1\ 2\ 3\ 4), \qquad V_3 = (4\ 3\ 2\ 1).$$

6. [15 %] Show that

   a) $\sum_{n=1}^{\infty} x^n(1-x)$ is pointwise convergent on $[0,1]$ but not uniformly convergent.

   b) $\sum(-1)^n x^n(1-x)$ converges uniformly.

7. [10 %] Determine the limit $\lim_{n\to\infty} \frac{e^{nx}-1}{e^{nx}+1}$ as a function of x.

8. [15 %] Let I_n be the $n \times n$ identity matrix and let $E_n = (e_{ij})$ be the $n \times n$ matrix with $e_{ij} = 1$ for $1 \leq i, j \leq n$. Define the set of $n \times n$ matrices

$$\mathcal{M} = \{M_{a,b} = aI_n + b(E_n - I_n), a, b \in \mathcal{R}\}.$$

   a) Show that members of $\mathcal{M}$ are commutable.

   b) For $a \neq b$, find c and d such that $(M_{a,b})^{-1} = M_{c,d} \in \mathcal{M}$.