參考用

1. a. State

(5%) (1) Spectral Decomposition

(5%) (2) singular Value Decomposition

(10%) b. Decompose $\begin{bmatrix} 37 & -2 & -24 \\ -2 & 13 & -3 \\ -24 & -3 & 17 \end{bmatrix}$ according to spectral decompostion.

2. Let A be a $n \times n$ symmetric matrix and $A^2 = A$. Show that

(5%) (1) every eigen value of A in either 0 or 1.

(5%) (2) if A is of full rank then $A = I$ (identity matrix)

(10%) (3) A is positive semi-definite and $A = \sum_{i=1}^{k} L_i L_i'$ where $L_1, \ldots, L_k$ are orthogonal vectors and $k = \text{rank}(A)$

(5%) (4) $\text{trace}(A) = \text{rank}(A)$

3. Let $f(\lambda) = \begin{cases} \frac{y^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \log y & \text{if } \lambda = 0 \end{cases}$

(5%) (a) Is $f(\lambda)$ continuous at $\lambda = 0$? why ?

(10%) (b) Find $f'(\lambda)$

4. Let f be a real-valued function, defined for (x, t) for $x \geq a$ and $\alpha \leq t \leq \beta$.

(10%) (1) what do we mean by

$$F(t) = \int_a^\infty f(x, t)dx \text{ exists ?}$$

(10%) (2) Using notation in (1), state Weierstrass M-test.

(10%) (3) Let $f(x, t) = e^{-xt}$, $x > 0$. Show that the convergence

$$\int_0^\infty e^{-tx}dx \text{ is uniform for } t > t_0 > 0.$$

(10%) (4) Show that $\log\left(\frac{\beta}{\alpha}\right) = \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx.$