國立中央大學107學年度碩士班考試入學試題

所別： 統計研究所 碩士班 不分組(一般生)　　共1頁　第1頁
統計研究所 碩士班 不分組(在職生)

科目： 數理統計

本科考試可使用計算器，廠牌、功能不拘　　*請在答案卷(卡)內作答

1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with mean $\theta > 0$.

   (a) (10 %) Find an unbiased estimator of θ based on $g(X_1, \ldots, X_n) = \sqrt[n]{X_1 X_2 \cdots X_n}$, the sample geometric mean.

   (b) (10 %) Is the unbiased estimator derived in (a) better than the sample (arithmetic) mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$? Give your reason.

2. (10%) Let $X_1, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance one. Let $p(x_1, \ldots, x_n)$ be the p-value in testing $H_0 : \mu = 0$ versus $H_1 : \mu = 1$, where x_i is the observed value of X_i, $i = 1, \ldots, n$. Find the distribution of $T = p(X_1, \ldots, X_n)$ when H_0 is true.

3. Let X and Y be two random variables.

   (a) (5 %) Prove or Disprove: If X and Y are independent, then X and Y are uncorrelated.

   (b) (5 %) Prove or Disprove: If X and Y are uncorrelated, then X and Y are independent.

4. (10 %) Suppose that $X_1, X_2, \ldots, X_{n_1}$, $Y_1, Y_2, \ldots, Y_{n_2}$, and $Z_1, Z_2, \ldots, Z_{n_3}$ are independent random samples from normal distributions with respective unknown means μ_1, μ_2 and μ_3 and common unknown variance σ^2. Derive the uniformly most powerful test for testing $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 < \mu_2$ at level α.

5. (10 %) Let Z be a random variable whose reciprocal, $1/Z$, has gamma distribution with shape parameter $1/2$ and scale parameter 2. Given $Z = z$, X has normal distribution with mean θ and variance $z\sigma^2$. What is the distribution of X? Show your work.

6. (10 %) Let $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$ chosen from a certain distribution and let Y_1, Y_2 and Y_3 be randomly selected (with replacement) from $\{x_1, x_2, x_3\}$. Find the variance of the maximum of Y_1, Y_2, Y_3.

7. Let $Y_1, Y_2, \ldots, Y_n$ be independent Bernoulli random variables with success probability $p \in [0, 1]$. Let $W = \sum_{i=1}^{n} Y_i$, and

$$T = \begin{cases} 1, & \text{if } Y_1 = 1 \text{ and } Y_2 = 0 \\ 0, & \text{otherwsie.} \end{cases}$$

   (a) (10 %) Find $E(T|W = w)$, the conditional expectation of T given $W = w$.

   (b) (10 %) Is $E(T|W)$ unbiased of $p(1-p)$? What can you conclude about it?

8. (10 %) Let $Y_1, Y_2, \ldots, Y_5$ be a random sample with probability density function

$$f(y) = (3y^2/\theta)\, e^{-y^3/\theta}, \; y > 0,$$

and zero otherwise. Construct an exact $(1-\alpha)100\%$ confidence interval for θ.

參考用