所別：統計研究所碩士班 不分組(一般生) 科目：基礎數學 共 1 頁 第 1 頁
統計研究所碩士班 不分組(在職生)
本科考試可使用計算器，廠牌、功能不拘 *請在試卷答案卷（卡）內作答

1. Evaluate

   a. (5%) $\lim_{x\to 0^+} x^{\sin x}$;

   b. (5%) $\lim_{x\to +\infty} \frac{\sqrt{2+x^2}}{x}$.

2. (10%) Determine all possible values of k such that the integral $\int_0^\infty \frac{x^k}{1+x^2}\,dx$ is finite.

3. (10%) Prove or disprove that the series $\sum_{n=1}^\infty \exp(-n^\epsilon)$ is convergent for all $\epsilon > 0$.

4. (10%) Verify for all $\tau > 0$ that

$$\int_0^\infty \exp\{-\frac{1}{2}\tau x^2\}\,dx = \frac{1}{\sqrt{\tau}}\frac{\sqrt{2\pi}}{2}. \quad (1)$$

5. (10%) Define $g(t) = \int_0^\infty \cos(tx)\exp(-\frac{x^2}{2})\,dx$ for $t \geq 0$ such that $g(t) > 0$. Work out $g(t)$.

6. (10%) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find a "positive square root" of A, that is, a matrix B such that $B^2 = A$ and B has positive eigenvalues.

7. (10%) Is the matrix $M = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$ diagonalizable?

8. (10%) Given an invertible $n \times n$ real matrix A and one column vector $\mathbf{u} \in \mathbb{R}^n$, consider $\mathbf{x} = A^{-1}\mathbf{u}$ and $\mathbf{y} = (A + \lambda\mathbf{u}\mathbf{v}^\top)^{-1}\mathbf{u}$ for any scalar λ and column vector $\mathbf{v} \in \mathbb{R}^n$ such that $A + \lambda\mathbf{u}\mathbf{v}^\top$ is also invertible. Determine whether $\mathbf{y}$ is parallel to $\mathbf{x}$.

9. Let $\mathbf{1}_n$ be the column vector of ones in $\mathbb{R}^n$. Given another $p-1$ vectors $\mathbf{x}_2, \ldots, \mathbf{x}_p$ of $\mathbb{R}^n$, consider a $n \times p$ matrix $X = [\mathbf{1}_n\ \mathbf{x}_2 \cdots \mathbf{x}_p]$ and define $P = X(X^\top X)^{-1}X^\top$. We assume $(X^\top X)^{-1}$ exists. Show

   a. (5%) $\mathbf{1}_n^\top P = \mathbf{1}_n^\top$;

   b. (5%) $H^2 = H$, where $H = P - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$;

   c. (5%) H is semi-positive definite;

   d. (5%) $p_{ii} \geq \frac{1}{n}$ for all $i = 1, \ldots, n$, where p_{ii} is the ith diagonal element of P.