所別：數學系碩士班 甲組　科目：高等微積分

(1) Find

(a) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$; (10%)

(b) $\lim_{n\to\infty} (1 + \frac{1+x+x^2+\cdots+x^n}{n})^n$, $|x| < 1$. (10%)

(2) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function; $f(f(x)) = x$ for all real x. Show that there exists $\xi \in \mathbb{R}$ such that $f(\xi) = \xi$. (10%)

(3) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that

$$\int_0^x [\int_0^u f(t)dt]du = \int_0^x f(u)(x-u)du. \quad (10\%)$$

(4) Let $a > 1$, $b > 1$, $f : \mathbb{R} \to \mathbb{R}$ is a bounded function and $f(ax) = bf(x)$ for all real x.

(a) Find $f(0)$. (5%)

(b) Show that f is continuous at 0. (10%)

(5) Let $f(x) = \int_0^\infty \frac{e^{-xt^2}}{1+t^2} dt, x \in (0, \infty)$. Show that $f(x)$ is uniformly convergent and differentiable on $(0, \infty)$. (10%)

(6) (a) Does series of functions $\sum_{n=1}^{\infty} e^{-nx} \sin(\frac{x}{n})$ converges uniformly on $[0, \infty)$? Give your proof. (10%)

(b) Determine whether the function $f(x) = \sum_{n=1}^{\infty} e^{-nx} \sin(\frac{x}{n})$ is uniformly continuous on $[0, 10]$. Give your reason. (5%)

(7) Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function. For each $n = 1, 2, 3, \cdots$, let $f_n(x) = \int_0^x f_{n-1}(t)dt$ for all $x \in [0,1]$. If for every $x \in (0,1)$, there exists an n such that x is an interior point of $f_n^{-1}(\{0\})$, prove that $f_0 \equiv 0$ on $[0,1]$. (10%)

(8) Determine whether in the system of equations

$$xv^2 + yu + x^2u^2 + y^3v = 0,$$

$$x^2u^6 + yv^3 + x^3y^2 + uv^2 = 0,$$

u and v are solvable in terms of x and y near $x = -1, y = 1, u = 1, v = -1$. If this can be done, compute $\frac{\partial v}{\partial x}(-1, 1)$ and $\frac{\partial u}{\partial y}(-1, 1)$. (10%)