所別：數學系碩士班 不分組 科目：線性代數

以下各題，只給答案，沒有說明，不給分

In the following, the symbol $\mathbb{R}$ denotes the field of real numbers as usual.

1. Let n be a positive integer. Let $A = (a_{i,j})$ be an $n \times n$ matrix whose entries $a_{i,j} = i+j-n$ for $i, j = 1, 2, \ldots, n$. Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T_A(x) = Ax$ for column vector $x \in \mathbb{R}^n$.

   (a) (15 分) Find the null space $N(T_A)$ and the range $R(T_A)$ of T_A by giving bases for $N(T_A)$ and $R(T_A)$ respectively.

   (b) (6 分) Prove or disprove that $\det(A) = 0$ if and only if $n \geq 3$.

   (c) (9 分) Prove or disprove that the charateristic polynomial $f(\lambda)$ of A is of the form $f(\lambda) = (-1)^n(\lambda^n - n\lambda^{n-1} + b\lambda^{n-2})$ for some $b \in \mathbb{R}$ such that $b \leq \frac{n^2}{4}$.
   Note. If you can not solve the problem for the general case, you can still get partial credit by verifying the case where $n = 4$.

2. (12 分) Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}, i = 0, \ldots, 3\}$ be the set of polynomials of degrees at most 3. Note that P_3 is a vector space over the real numbers. Let $D : P_3 \to P_3$ be the linear operator defined by $D(f(x)) = (x+2)f'(x)$ for all $f(x) \in P_3$. Is D a diagonolizable operator? Explain your answer. If the answer is yes, give an ordered basis β of P_3 such that $[D]_\beta$ is a diagonal matrix.

3. Let V, W be vector spaces over a common field F and let $\mathcal{L}(V, W)$ denote the set of all linear transformations from V to W. Assume that V is finite dimensional. Fix an operator U on W (that is, $U \in \mathcal{L}(W, W)$). Let $T_U : \mathcal{L}(V, W) \to \mathcal{L}(V, W)$ be the linear transformation $T_U(S) = US$ (composition of U and S) for $S \in \mathcal{L}(V, W)$. Let $f(\lambda)$ be the charateristic polynomial of U. Prove the following statement or disprove it by giving counterexamples.

   (a) (6 分) $f(T_U) = 0$, the zero transformation on $\mathcal{L}(V, W)$.

   (b) (12 分) Assume that W is of finite dimensional with $\dim W = m$. Let $g(\lambda)$ be the characteristic polynomial of T_U, then $g(\lambda) = f(\lambda)^m$.

4. Let n be a positive integer and let $V = M_{n\times n}(\mathbb{R})$. Define $\langle X, Y \rangle = \mathrm{Tr}(Y^tX)$ for $X, Y \in V$, where Y^t denotes the transpose of Y and $\mathrm{Tr}(A) = \sum_{i=1}^n A_{i,i}$ denotes the trace of a matrix A. Fix an $n \times n$ matrix A. and define the linear operator $T_A : V \to V$ by $T_A(X) = AX$ for any $X \in V$.

參考用

注意：背面有試題

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(a) (10 分) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.

(b) (7 分) What is the adjoint operator T_A^* of T_A? Explain your answer.

(c) (8 分) Let $W = \{X \in V; \mathrm{Tr}(X) = 0\}$. Compute the orthogonal complement $W^\perp$ of W by giving an orthonomal basis for $W^\perp$. What is $\dim W^\perp$? Explain your answer.

5. (15 分) Let $L_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{i,j}\, x_j,\ a_{i,j} \in \mathbb{R}$ for $i = 1, \ldots, m$. Let

$$W = \{(b_1, \ldots, b_n) \in \mathbb{R}^n; L_i(b_1, \ldots, b_n) = 0, \ i=1,2,\ldots,m\}$$

be the subspace of $\mathbb{R}^n$ determined by the common zeros of the linear functionals $L_1, L_2, \ldots, L_m$. Let $f(x_1, x_2, \ldots, x_n)$ be a linear functional such that $f(b_1, \ldots, b_n) = 0$ for all $(b_1, \ldots, b_n) \in W$. Prove or disprove that there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ such that $f = \sum_{i=1}^{m} \lambda_i L_i$

參考用