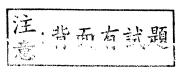
| 國立中 | 央大學九十二學年度 | 碩士班考試 | 入學招生計寫出 | 41. 7 | |
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| 系所別: | | • | 加工机规范 | 共 <u>之</u> 頁 | 第頁 |
| | 数學系 | 科目: | 抽象代數 | | |

以下各題,只給答案,沒有說明,不給分

In the following, the symbols $\mathbb Q$ and $\mathbb C$ denote the fields of rational numbers and complex numbers as usual.

- 1. Determine whether or not the following statements is correct. Explain your answers.
 - (a) (4分) For any given positive integer n there exists a finite group of order n.
 - (b) (6分) For any given positive integer n there are only finitely many non-isomorphic groups of order n.
- 2. Let G be a non-Abelian (non-commutative) group and let A be a cyclic group. Assume that G has an action * on A which satisfies (1) $(\sigma\tau)*$ $a = \sigma*(\tau*a)$ for all $\sigma, \tau \in G$ and all $a \in A$; and (2) $\sigma*(ab) = (\sigma*a)(\sigma*b)$ for $\sigma \in G$ and $a, b \in A$.
 - (a) (8分) Show that the action * induces a group homomorphism from G to $\operatorname{Aut}(A)$ where $\operatorname{Aut}(A)$ is the automorphism group of A.
 - (b) (4分) Show that there exist a non-trivial normal subgroup H of G of finite index (that is, $\{e\} \neq H \triangleleft G$ and [G:H] is finite) so that $\sigma * a = a$ for all $\sigma \in H$ and all $a \in A$.
 - (c) (4 \Re) Suppose that A is an infinite cyclic group. Show that the index [G:H] of H in G is either one or two.
- 3. Let p be a prime number and let S_p be the symmetric group on p symbol
 - (a) (10 分) Determine the number of p-Sylow subgroups of S_p (Hipp first show that S_p has (p-1)! elements of order p).
 - (b) (8分) What are the numbers of p-Sylow subgroups of S_{p+i} for any i such that $1 \le i \le p-1$? You need to explain your answer.
- 4. Let R be a finite ring.
 - (a) (7分) Can R be an integral domain if R has order |R|=36? Why?
 - (b) (7分) What should be a necessary condition for the order of R so that R can be an integral domain? Explain your answer.
- 5. Let F be a field. Let $f_1(x), f_2(x), \ldots, f_n(x) \in F[x]$ be polynomials which are not all zero. The G.C.D. of $f_1(x), f_2(x), \ldots, f_n(x)$ is defined to be the monic polynomial of



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maximal degree among common divisors of $f_1(x), f_2(x), \ldots, f_n(x)$. Denote the G.C.D. by $gcd(f_1(x), f_2(x), \ldots, f_n(x))$. Prove

$$\gcd(f_1(x), f_2(x), \dots, f_n(x)) = \sum_{i=1}^n a_i(x) f_i(x)$$
 for some $a_i(x) \in F[x], i = 1, \dots, n$

by completing the following steps.

- (a) (12 %) Show that F[x] is a principal ideal domain.
- (b) (8%) Let \mathcal{A} be the ideal generated by $f_1(x), f_2(x), \ldots, f_n(x)$. Show that \mathcal{A} is also generated by $\gcd(f_1(x), f_2(x), \ldots, f_n(x))$ and conclude that

$$\gcd(f_1(x), f_2(x), \dots, f_n(x)) = \sum_{i=1}^n a_i(x) f_i(x).$$

- 6. (10%) Let a, b be relatively prime non-zero integers such that at least one of a, b is not ± 1 . Assume that both a and b are square free integers. Is it true that the polynomial $ax^m b$ is irreducible in $\mathbb{Q}[x]$ for every positive integer m? Explain your answer. Note. An integer n is called square free if it has that property that for prime p with $p \mid n$ then $p^2 \nmid n$.
- 7. (12%) Let P(x) be an irreducible polynomial in $\mathbb{Q}[x]$. Let $r \in \mathbb{C}$ be a root of P(x).

$$\alpha = \frac{a_n r^n + \dots + a_1 r + a_0}{b_m r^m + \dots + b_1 r + b_0} \text{ where } a_i, b_j \in \mathbb{Q} \text{ and } b_m r^m + \dots + b_1 r + b_0 \neq 0.$$

Prove or disprove that there exists a polynomial $f_{\alpha}(x) \in \mathbb{Q}[x]$ such that $\alpha = f_{\alpha}(r)$.

