國立中央大學八十六學年度碩士班研究生入學試題卷

- **SHOW YOUR WORK & GOOD LUCK !**
- Use Table 1 & Table 2 if necessary

1. (10%) Let $Y_1, Y_2, \ldots, Y_6$ be a random sample from a normal population with a mean μ and a variance of σ^2. Let $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$ and let $U = \sum_{i=1}^{5}(Y_i - \bar{Y})^2$. What is the distribution of $\frac{2Y_6 - 2\mu}{\sqrt{U}}$? Why ?

2. (20%) If $Y_1, Y_2, \ldots, Y_n$ denotes a random sample from an exponential distribution with a mean of θ.
   (a). (14%) Find 2 sufficient statistics for θ. Justify your answers.
   (b). (6%) Which sufficient statistics in (a) is better in terms of sufficiency ? State your reason.

3. (15%) An urn contains θ white balls and $N - \theta$ black balls. A sample of n balls is to be selected without replacement. Let Y denote the number of white balls in the sample. Find the method of moments estimator of θ.

4. (20%) Suppose $Y_1, Y_2, \ldots, Y_n$ denotes a random sample from the Weibull density function, given by

$$f(y) = \begin{cases} \frac{2y}{\theta} e^{-\frac{y^2}{\theta}}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

   (a). (10%) Find the minimal sufficient statistics for θ. Justify your answer.
   (b). (10%) Find a MVUE for θ. That is to find a minimum variance unbiased estimator for θ. Justify your answer.

5. (15%) Two methods for teaching reading were applied to two randomly selected groups of elementary schoolchildren and compared on the basis of a reading comprehension test given at the end of the learning period. The sample means $\bar{y}$ and sample variances s^2 computed from the test scores are shown in the accompanying table. Find a 90% two-sided confidence interval for $(\mu_1 - \mu_2)$. Provided that the scores are normally distributed with mean μ_1, μ_2 and the same variance. Note that if $Y_1, \ldots, Y_n$ is a random sample then sample mean $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y - \bar{Y})^2$.

Statistics	Method 1	Method 2
No. Children in group	8	9
$\bar{y}$	63	67
s^2	50	72

6. (20%) Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ for $i = 1, \ldots, n$ be n random variables. Let $S_{xY} = \sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})$ and $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then $\hat{\beta}_1 = \frac{S_{xY}}{S_{xx}}$ and $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{x}$ are the least squares estimators for β_1 and β_0, respectively. Suppose $\varepsilon_1, \ldots, \varepsilon_n$ is a random sample from the normal distribution with mean 0 and variance σ^2.
   (a). (10%) Find the distribution of $\hat{\beta}_1$.
   (b). (10%) Suppose we fit the model $Y = \beta_0 + \beta_1 x + \varepsilon$ to n data points $(x_1, y_1), \ldots, (x_n, y_n)$. Also, suppose that $\sigma^2 = 1$. Consider the test of the hypothesis $H_0 : \beta_1 = 0$ against $H_a : \beta_1 \neq 0$. Give the rejection region of above test in terms of the n data. Use a significance level $\alpha = .05$.

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