國立中央大學八十五學年度碩士班研究生入學試題卷

所別：數學研究所　不分組　　科目：統計　　共 2 頁 第 1 頁

注意事項：

- 請列出計算過程，僅有答案，不予計分。
- 如需要，可用所附表格求值。
- 請務必將題號標示清楚。可不按順序作答。

1. Let X_1, X_2 be a random sample from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}e^{-x/\theta}, & 0 < x < \infty,\ 0 < \theta < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) What is the distribution of X_1/X_2? Give your reason. (10%)
   (b) What is the distribution of $X_1/(X_1 + X_2)$? Give your reason. (10%)

2. Let $X_i \sim \chi^2(r_i)$ (i.e. X_i is chi-square distributed with r_i degrees of freedom), $i = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are independent.
   (a) Show that $X_1 + X_2 + \cdots + X_n \sim \chi^2(r_1 + r_2 \cdots + r_n)$. (10%)
   (b) Let $Y_n \sim \chi^2(n)$. Show that, as $n \to \infty$,

$$\frac{Y_n - n}{\sqrt{2n}} \to N(0,1)$$

   in distribution. (Hint: Use part (a) and the Central Limit Theorem.) (10%)
   (c) Suppose a machine in a factory produces steel rods of length Z, where Z is normally distributed with mean μ of 7 inches and variance σ^2 of 0.3. The cost C of repairing a rod that is not exactly 7 inches in length is given, in dollars, by $C = 5(Z - \mu)^2$. If 50 rods with independent lengths are produced in a given day, estimate the probability that the cost for repairs for that day exceeds \$90. (10%)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \theta^2 x e^{-\theta x}, & 0 < x < \infty,\ 0 < \theta < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Show that $Y = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ. (10%)
   (b) Find the unique UMVE (unbiased minimum variance estimator) of θ. (10%)

4. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two independent random samples from normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, where μ_1, μ_2, σ_1^2, and σ_2^2 are all unknown. Let $\bar{X} = n^{-1}\sum_{i=1}^n X_i$, $S_1^2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $\bar{Y} = m^{-1}\sum_{i=1}^m Y_i$, $S_2^2 = m^{-1}\sum_{i=1}^m (Y_i - \bar{Y})^2$. Define

$$F = \frac{\frac{mS_2^2}{\sigma_2^2}/(m-1)}{\frac{nS_1^2}{\sigma_1^2}/(n-1)}.$$

   (a) What is the distribution of F? Give your reason. (10%)
   (b) Given $m = 13$, $n = 16$, and two observed values s_1^2 and s_2^2, use part (a) to construct a 95% confidence interval for σ_1^2/σ_2^2. (10%)

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$. Find the m.l.e. (maximum likelihood estimator) of $e^{-2\theta}$. (10%)

The Normal Distribution

$$\Pr(X \le x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$[\Phi(-x) = 1 - \Phi(x)]$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999

The F-Distribution

$$\Pr(F \le b) = \int_{0}^{b} \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2 - 1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1 + r_2)/2}}\, dw$$

$\Pr(F \le b)$	r_2	r_1 = 1	2	3	4	5	6	7	8	9	10	12	15
0.95	1	161	200	216	225	230	234	237	239	241	242	244	246
0.975		648	800	864	900	922	937	948	957	963	969	977	985
0.99		4052	4999	5403	5625	5764	5859	5928	5982	6023	6056	6106	6157
0.95	2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
0.975		38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4
0.99		98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
0.95	3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70
0.975		17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.3	14.3
0.99		34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9
0.95	4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86
0.975		12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66
0.99		21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2
0.95	5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62
0.975		10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43
0.99		16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72

$\Pr(F \le b)$	r_2	r_1 = 1	2	3	4	5	6	7	8	9	10	12	15
0.95	6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94
0.975		8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27
0.99		13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56
0.95	7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51
0.975		8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57
0.99		12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31
0.95	8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22
0.975		7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10
0.99		11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52
0.95	9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
0.975		7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77
0.99		10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96
0.95	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85
0.975		6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52
0.99		10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56
0.95	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62
0.975		6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18
0.99		9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01
0.95	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40
0.975		6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86
0.99		8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52